on one side of the equation:

$$\begin{aligned} y &= 2e^{x+1} && \text{our initial equation} \\ \frac{y}{2} &= e^{x+1} && \text{divide both sides by 2} \\ \ln\frac{y}{2} &= x + 1 && \text{take the natural log of both sides} \\ \ln\frac{y}{2} - 1 &= x && \text{subtract 1 from both sides} \end{aligned}$$

Now that x is by itself, to get the inverse function we just use the side of the equation that has y, but we substitute in the variable x for y, giving us:

$$f^{-1}(x) = \ln\frac{x}{2} - 1$$

One problem for inverse functions is that, for most functions, their inverse is not a true function. Remember that a function produces a single value from its inputs. However, if the graph of your original function goes both up and down, then the inverse of that function will have *multiple* values for a given x value.

For instance, take the function $f(x) = x^2$. This is a true function—for any given x there is exactly one result. What is the inverse? The inverse of this function is $f^{-1}(x) = \sqrt{x}$. However, square roots can be positive or negative, so this is more accurately presented as $f^{-1}(x) = \pm\sqrt{x}$. Thus, for a given x value, there might be as many as two distinct values.

With higher exponents, there are more possible values in the inverse function. Some functions, such as sin and cos, have inverses which have infinitely many results for any given value. For these, the inverse functions are usually given a pre-defined fixed range of possible results.

For the most part, this is not a terrible situation, but it is good to keep in mind that even if $f(x)$ is a true function, $f^{-1}(x)$ might not be, or it may have to be used with special qualifications.

4.7 Inventing Functions and the Magic of Substitution

In section 4.5 we looked at how to use functions, analyze them, and combine them into new functions. In this section, we will look at how to take an existing equation and *invent* new functions to help us look at equations under new lights.

Let's look at the following equation:

$$\sin^2(x) + 2 \cdot \sin(x) + 1 = 0$$

How might this be solved?

In order to solve this, I want you to think about different types of equations you have encountered in your time doing mathematics. Does it remind you of anything, even if it isn't exact? If you notice, the first term is squared, the second term is to the first power, and the third term is a constant. What sort of equation kind of looks like that?

If you said "quadratic equation," you would be correct. If you didn't, take a close look, and see if you can at least see the resemblance between a quadratic equation and the equation above. For a reminder, the basic form of a quadratic equation is

$$ax^2 + bx + c = 0 \tag{4.1}$$

where a, b, and c are constants. To solve such an equation for x, you use the quadratic formula. For reference, the quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4.2}$$

Now, there is one problem—the equation doesn't exactly match the quadratic equation. Each time we are supposed to have an x, we have a $\sin(x)$ instead. So what do we do?

Once you recognize that every place where we wanted to have x that we instead have $\sin(x)$, then we can create a function to *convert* one to the other.

So, what we wanted was x, but what we got instead was $\sin(x)$. Therefore, we will create a new variable, u. We will then say that:

$$u = \sin(x)$$

Now, we have decided, using our own creative power, that we want to have a new variable u to be equal to $\sin(x)$. Now, every place where we have $\sin(x)$, we can replace it with u, based on our definition.

So, replacing $\sin(x)$ with u gives us this equation:

$$u^2 + 2u + 1 = 0$$

This now actually *is* a quadratic equation! The only difference is that it uses the variable u instead of x. However, the rules of algebra are the same no matter what we name our variables.

Therefore, using the quadratic formula, we can find the answer:

$$\begin{aligned} u &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\ &= \frac{-2 \pm \sqrt{4 - 4}}{2} \\ &= \frac{-2}{2} \\ &= -1 \end{aligned}$$

So, we have established that $u = -1$. Are we finished? Not quite. Remember that the oiginal problem was to find x, not u. However, when we did the transformation, we also built an equation that relates x and u, namely $u = \sin(x)$. Therefore, since they are equivalent, we can substitute the other way as well, putting $\sin(x)$ back in for u and solve the equation:

$$\begin{aligned} u &= -1 \\ u &= \sin(x) \\ \sin(x) &= -1 && \text{since both of these are equal to } u \text{ they are equal to each other} \end{aligned}$$

Now, the inverse operation of $\sin(x)$ is $\arcsin(x)$ (also known as $\sin^{-1}(x)$). If we apply that to both sides, it will undo the $\sin(x)$ on the left:

$$\begin{aligned} \arcsin(\sin(x)) &= \arcsin(-1) \\ x &= \arcsin(-1) && \text{since arcsin is the inverse of sin, we just get } x \\ x &\approx -1.5708 \end{aligned}$$

If you got -90, that is because your calculator is in degree mode instead of radian mode. In calculus, we usually calculate using radians instead of degrees unless otherwise specified.

So, let's go back through our steps. First, we thought about if our strange equation was related in any way to a more standard equation that we already knew. Once we figured out that it was similar to a quadratic equation, we then looked for a way to transform the equation into a form that we could use the quadratic formula for. We found out that the big problem standing in our way was that the equation was using $\sin(x)$ instead of x in each term. Therefore, we decided to make a substitution, substituting u (which was a function of x) for $\sin(x)$. This gave us the form that we wanted in order to solve it using the quadratic formula. After we solved for u, we then needed to get back to x. We used our original equation relating u and x to get our answer for u in terms of x. Then, we solved for x.

It sounds like a lot of steps, but, in brief, we invented a new variable which we could substitute for another variable in the equation to make it solvable. You have to be careful when doing that, however, because you have to make sure that the substitution actually can replace all of the instances of x in the equation. If you don't, then you have just *increased* the number of variables in your equation. That isn't necessarily wrong, but it usually is counterproductive.

For instance, if the equation had instead been $\sin^2(x) + 2x + 1 = 0$, then the substitution would not have worked. You would have wound up with $u^2 + 2x + 1 = 0$, which is still not in a solvable form. We just have more variables.

Inventing variables to substitute for functions in an equation is a very powerful tool, but you do have to be sure you are using it right.

4.8 Multiple Variables

A function can be a function of more than one variable. For instance, the distance a person travels when moving depends on both their speed and the amount of time they spend traveling. Therefore, we can write a function to represent this:

$$d(\text{speed}, \text{time}) = \text{speed} \cdot \text{time}$$

This indicates that the function $d()$ (i.e., distance) is based on two parameters—speed and time. Therefore, in this function, *both* speed and time are independent variables, and the distance travelled is the dependent variable.

Often times, in functions of two variables, x and y are both the independent variables, and z becomes the dependent variable. These can be represented by graphs in three dimensions. For instance, take the function $f(x, y) = 3xy$. If we want the value of $f(4, 5)$, we would get $3 \cdot 4 \cdot 5 = 60$. Therefore, our dependent variable (which we would likely graph as z) would be 60.

Note that functions can *also* have more than one dependent variable.