In any case, I just wanted to point out how powerful it is just to put a name on a particular problem. This is often true in life, too. Sometimes if we run into difficulties, just naming what it is that is happening will help us understand the issue in the future, and, eventually, we will figure out how to solve the problem.

26.5.3 Infinite Series Representation

The actual point of this chapter is to show how we can solve the integral by overcoming the other assumption—that we use only a finite number of terms. Just like in Chapters 25 and 26, we were able to convert a function that we couldn't express in terms of a finite number of polynomials into an infinite series of polynomials.

Why does that matter to us in trying to integrate $\cos(x^2)\,\mathrm{d}x$?

Think about a polynomial. Each individual term in a polynomial is just a constant multiplier and a variable raised to a power. Nothing could be easier to solve for an integral! Also, the terms of a polynomial are all separated by addition. This means that there is nothing special about how they combine! If we can convert a function into an infinite series of polynomial terms (especially one that can be written using summation notation), then all we have to do to find the integral is to integrate the single term being summed![6] So, let's start by finding a Maclaurin series for $\cos(x^2)$. Equation 25.8 says that:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Remember that in this formula, $f^{(n)}(0)$ refers to the nth derivative of f evaluated at 0. So, if we use a_n to refer to the coefficient of each term of the series, we can expand the series to say:

coefficient	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
a_0	$\cos(x^2)$	1	1
a_1	$-\sin(x^2)\,2x$	0	0
a_2	$-2\sin(x^2) - 4x^2\cos(x^2)$	0	0
a_3	$8x^3\sin(x^2) - 12x\cos(x^2)$	0	0
a_4	$48x^2\sin(x^2) - 12\cos(x^2) + 16x^4\cos(x^2)$	-12	$-\frac{1}{2}$
...			

This goes on quite a while. It is long and painful enough that we won't represent it here, but I did want to make sure you recognized that it could be done.

Eventually, you get to the point that you can see that the formula becomes:

$$\cos(x^2) = 1 + -\frac{x^4}{2} + \frac{x^8}{24} + -\frac{x^{12}}{720} + \frac{x^{16}}{40,320} + \ldots$$

This can be reduced to a summation notation, but it is somewhat complicated to generate that from the expansion above.

[6] One student, when hearing that infinity comes to the rescue for performing impossibly hard integrals, made the comment that this procedure reminds her of the verse in the Bible that says, "with man this is impossible, but with God all things are possible" (Matthew 19:26). I don't know if that exactly fits (an infinite sequence isn't God), but it does certainly resonate. The endless (i.e., infinite) unfailingness of God's nature is what allows us to surpass the limits of our finitude. If we put our trust in mathematics for generalizing the effectiveness of an infinite series to an infinite number of terms, how much more should we trust in God to add up our limited actions to serve in His infinite purposes? Thinking about infinity naturally leads to thinking about God, and, while we should be careful to not necessarily equate the properties of the two prematurely, I do think that practice in thinking about the infinite certainly helps us in thinking more clearly about God.