Chapter 25

Modeling Functions with Polynomial Series

In this chapter we will look at ways of modeling special functions using polynomials with an infinite sequence of terms. We will also discuss a general approach for solving impossible problems.

25.1 Modeling Complicated Functions with Simple Tools

When people first learn polynomials, they get used to being able to express curves in terms of taking a variable and then manipulating it through squaring, cubing, and so forth. When you first start doing this, it seems that the entire universe of functions can be modeled in this way. Part of that is because, in fact, these tools are very powerful, but part of it is a selection effect—your teachers were only giving you problems in which these were the proper tools.

Therefore, when you get to learning about sine and cosine, it sometimes surprises people to learn that, here we have curves, but these curves cannot be modeled through a polynomial. It takes an entirely new type of function to model these curves!

As it turns out, there is a huge variety in different types of functions that cover different types of curves. There are trigonometric functions, exponential functions, hyperbolic functions, and all sorts of other types of functions. We get even more classes of functions if we don't limit ourselves to curves, add in dimensions, etc.

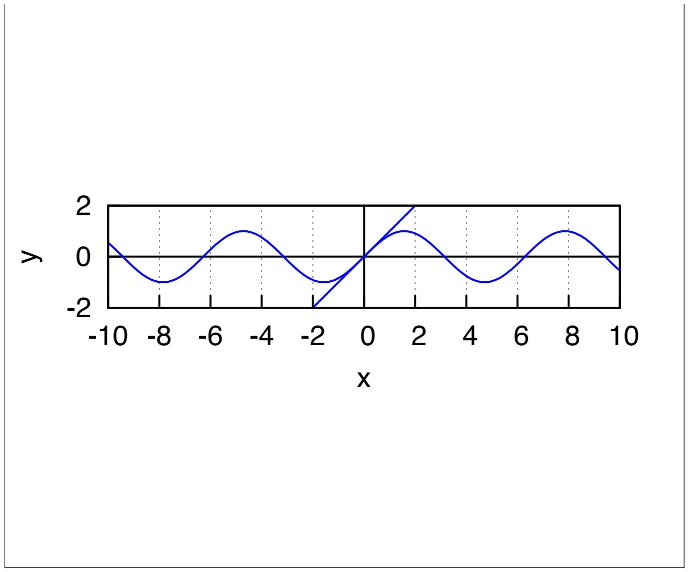
However, as we have learned in Calculus, it is often beneficial to model complicated things with simple tools. When applying the integral geometrically, for instance, we just used basic formulas of simple shapes to help us figure out the formulas for more complicated shapes. The area of a rectangle, the volume of a box, and the volume of a cylinder were all we needed to know in order to find areas and volumes of all sorts of complex shapes. Using simple tools make complicated things much easier to analyze.

In Chapters 18 and 20, we figured out that starting off with a simple and straightforward estimation procedure that can be improved is oftentimes the first step to getting an exact answer. In this chapter, we are going to focus on approximating functions like $\sin(x)$, $\cos(x)$, and e^x. You might be used to just plugging in numbers into a calculator to get the answers to these functions, but what procedure does the calculator use?

It turns our that the calculator can only approximate the values for these types of functions, but, it is "close enough" that you would never notice with the number of digits that the calculator displays.

Now, we can approximate functions using just about anything. If we wanted to, we could approximate a function with a straight line. It will be wrong more than it is right, but we can make an approximation that works for a certain set of points.

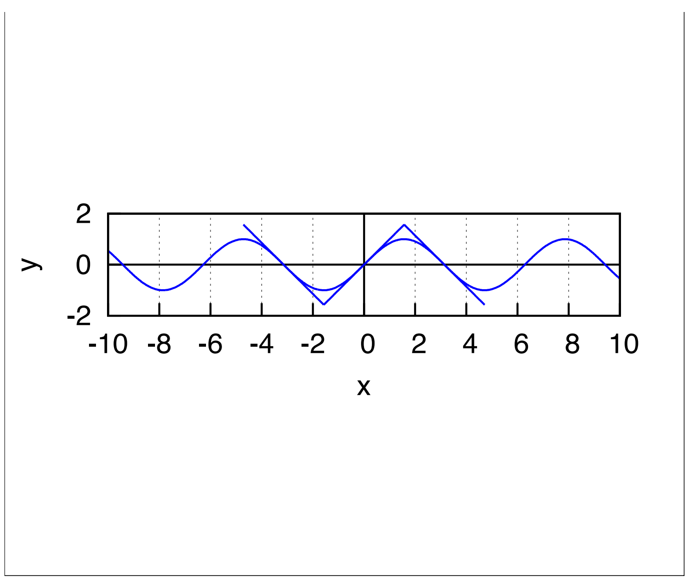
Here is $\sin(x)$ approximate using a straight line:



As you can see, the line actually approximates the graph pretty well from about -0.5 to 0.5.

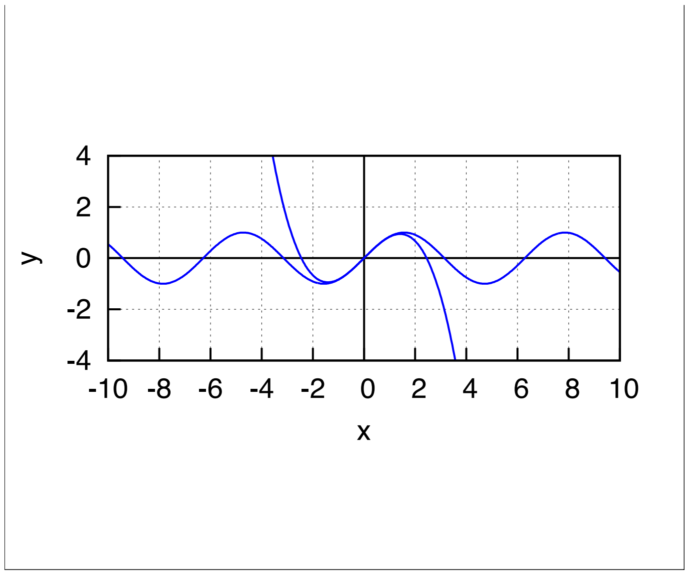
Now, if we take that approximation and increase the number of lines, our approximation of the function becomes closer and closer to reality. That is, we can make closer approximations of functions using two lines rather than with one. As we add more lines, we can get our approximation closer and closer to the "true" function.

Here is $\sin(x)$ approximated using multiple lines:

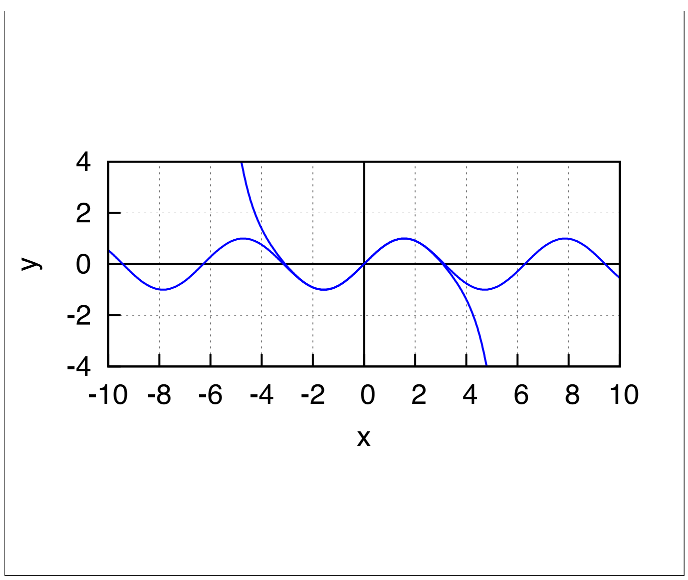


we can add more lines in-between and get a better and better approximation. However, the problem with lines is that lines are not smooth. In calculus, we like smooth functions because you can analyze slope easier.

Therefore, instead of using straight lines to model function behavior, we can use polynomials. Here is $\sin(x)$ modeled with a cubic equation (the equation, for reasons you will see later, is $y = x - \frac{1}{6}x^3$):



As you can see, the model works pretty well from about -1.5 to 1.5. The model becomes even closer as we add more terms. This next equation is $y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$, and it works well from about -2.5 to 2.5:



As we add terms to our polynomial, the domain for which it serves as a valid approximation of $\sin(x)$ increases.

25.2 Polynomials, Their Importance, and Their Limitations

So why do we want to limit ourselves to a polynomial model of a function like $\sin(x)$?

Well, in modern life, we want to do everything with computers. Computers, however, do not intrinsically understand functions like sine and cosine. You may have a cosine button on your calculator, but your calculator doesn't *actually* know how to calculate the cosine of a number. It only knows how to *approximate* the cosine of a number.

Calculators are computers, and computers are pretty limited in the type of mathematical operations they can perform. They fool us all the time through clever programming, but at the end of the day, the core set of features of computers are pretty much restricted to adding, subtracting, multiplying, dividing, comparing, branching (choosing to do some operation or set of operations based on a previous comparison), and moving numbers around.

Modern processors might throw in a few more operations, but they are pretty tied down to operations on this level. Every other operation that a computer does, it has to reduce to these operations. Exponentiation to an integer power can be achieved by repeated multiplication (and, in fact, even multiplication is achieved by mere repeated addition!).

Thus, in actuality, since computers can only add, subtract, multiply, divide, and do integer exponents, computers can *only* actually process polynomials. They can't do anything else *at all*.

Therefore, if we want to model various different kinds of functions in a computer (trig functions, hyperbolic functions, exponentials, etc.), then we have to be able to approximate these functions with polynomials.

Now, one of the interesting features of modeling functions with polynomials is that, if you restrict the domain (i.e., the allowable x-coordinates) that you are looking at, you can make a polynomial closer and closer to the target function by adding more and more terms to the polynomial. We saw this in the previous section with our approximations of the $\sin(x)$ function.

Depending on the function you are trying to model, your polynomial might wind up with too many or too few humps *elsewhere* (i.e., outside of your restricted domain), but within that restricted domain you can get closer and closer to the "real" function by adding terms to polynomials, and the domain for which your polynomial serves as an accurate measurement will increase as well.

Note that, in addition to modeling known functions (like $\sin(x)$ and e^x), polynomial modeling is also used a lot in engineering when we want to model functions where the actual physical relationships are not known, but we have a graph that we want to match up with. We can simply create a polynomial that gives us the right shape within a certain boundary, and then only use the polynomial equation within that boundary. This provides a smooth curve that can be used for approximating the "true" underlying function, whatever it actually is.

Unfortunately, it is impossible to model the complete $\sin(x)$ function exactly using polynomials.

25.3 Doing the Impossible

Of course, just because something is impossible doesn't mean we aren't going to do it anyway! After all, this is calculus class!

So, how might you model a non-polynomial *perfectly* using polynomials? How do you do the impossible?

It turns out that when anybody tells you that something is *impossible*, what they usually mean is "given a set of assumptions that I may not even realize I am making, this is impossible." The key to solving impossible situations is to learn to identify the unstated assumptions. Once you realize the sorts of things that people are unknowingly assuming, whole worlds open up for you. You find out that many times people may be correct in what they are saying, but they are presuming a whole set of assumptions that their listeners might not hold to.[1]

In philosophy, these unstated assumptions are known as *ceteris paribus* clauses, which means "all else being equal" in latin. The goal is to find out what in this universe of assumptions is causing problems, and then try to imagine the world without those assumptions. Doing so will help you figure out which assumptions are needed, and, more importantly, *when* are they needed.

I've often seen people argue about things, where the arguers on both sides were technically correct in their statements, but the statements only applied within certain sets of assumptions. Where they actually differed was not on their statements, but on what sets of assumptions (i.e., what ceteris paribus clauses) made their statements true. A better discussion would have been had if the arguers would have instead talked about which sets of underlying assumptions better represented reality, or more specifically the situation which they were discussing.

In this chapter, we are focusing on doing the impossible. To do the impossible, you first have to identify the set of statements that make something impossible. Once you do that, you find out which of the assumed statements can be removed to make the impossible possible.

When we say that "it is not possible to model a cosine function with a polynomial," what we are actually saying is "it is not possible to model a cosine function with a polynomial *using a finite number of terms*." We just assume, quite reasonably, that since we generally have to write down our equations, any equation we come up with will have to have a finite number of terms, because we couldn't possibly write down an infinite number of them. However, if we allow ourselves an *infinite* number of terms, then all of a sudden it becomes possible to model additional classes of functions with only polynomials.

25.4 Modeling with an Infinite Number of Terms

If we have a polynomial with an infinite number of terms, we first have to stop and think of what that will look like. We are going to take several stabs at writing this down, as the first one will not be the best way to do so.

So, how do we write down a polynomial generally? Usually, we write something down like:

[1] Arthur C. Clarke formulated several adages known as Clarke's Three Laws. The first one is relevant here—"When a distinguished but elderly scientist states that something is possible, they are almost certainly right. When they state that something is impossible, they are very probably wrong." While I am a big proponent of doing impossible things (and one of my goals this chapter is to make sure you know how to do them!), I find this formulation unhelpful.

I have seen experts be equally wrong on both sides of the table. Sometimes they say that something is possible and they are wrong. Sometimes they say that something is impossible and they are wrong. I think what he was getting at is that an expert probably knows where the current trends in his own field will lead, but does not have the same insight about new trends that are just being explored. They will have a tendency to regard the trends that they have dealt with their whole lives as being the "real" trend, and discount the new trends as being just a bunch of kids who don't know what they are doing. I think that is a true statement of human nature, but I think it gives us no benefit to knowing who is correct. Young upstarts can be just as correct or incorrect as old fogeys.

I think the key is simply being philosophical enough to recognize the assumptions that different people are making, where those assumptions lead, and then, in the last place, to determine which assumptions are true. I think we make a mistake to presume that the new assumptions are better or worse than the old ones. What we must do, is to examine everything with great care, and hold on to the good stuff (1 Thess 5:21). We shouldn't dismiss things because they are new, and we shouldn't consider anything to be outmoded just because it is old. That, fortunately or unfortunately, means that each one of us is responsible for a whole lot of philosophical legwork. However, as I pointed out in Chapter 2, for the mathematician, the first step of mathematics is philosophy. So learning to think clearly is something you already need to be learning to do!

$$y = ax^2 + bx + c$$

In this equation, a, b, and c are the *coefficients*. We can expand this idea to a higher-order polynomial by doing:

$$y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

We can keep expanding this out, but soon we will run out of letters. So, instead of using different letters for coefficients, we will simply number them. Since they are constant, we will represent them with a C, and so the first coefficient will be C_0, the next C_1, etc. However, since this is an infinite-term polynomial, since we normally write the highest power x on the left, that will be x^∞.

Therefore, the first way we will try to write the general form of the polynomial will look something like the equation below, with C_n representing a particular constant:

$$\cos(x) = C_0x^\infty + C_1x^{\infty-1} + C_2x^{\infty-2} + C_2x^{\infty-3} \ldots C_{\infty-2}x^2 + C_{\infty-1}x + C_\infty \tag{25.1}$$

So, we can define our function as an infinite set of constants attached to a polynomial. However, the way that it is written gets in the way of really manipulating it. The reason is that you have the constants starting at subscript zero, but the powers start at infinity. Since the subscripts of the constants are arbitrary, we can actually rewrite this with the subscripts starting at infinity and moving down, just like the powers. That looks like this:

$$\cos(x) = C_\infty x^\infty + C_{\infty-1}x^{\infty-1} + C_{\infty-2}x^{\infty-2} + C_{\infty-3}x^{\infty-3} \ldots C_2x^2 + C_1x + C_0 \tag{25.2}$$

Here, the subscripts match the powers. This doesn't do a whole lot, but it certainly helps us think about the series easier. It is very clear that C_{200} goes with the x^200 power, and so forth. Even C_0 follows this rule, since x^0 is just one. We just don't bother to write x^0 since it doesn't matter.

The problem, though, is that the sequence, as written, has two ends. It has a beginning, at infinity (which is certainly hard to conceive of, and doesn't really exist in the way that the other numbers do), and then has an ending at C_0. Can you think of another way to write this that might be even easier to manage?

Think of it this way—while we normally write polynomials with the highest power first (as it is in the attempts above), we will wind up being able to do interesting things if we turn it around and write the highest power last, like this:

$$\cos(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots + C_\infty x^\infty \tag{25.3}$$

In this form, the subscripts of the constants match the power of x as before. But, since we started at the low powers, the only ones we have to write down are the finite powers. The "infinity" is implied by the "..." at the end. It just goes and goes, and we don't even have to name infinity explicitly.

This is known as a **polynomial expansion** of the cosine function.

Because we the subscripts and the powers line up so nicely, it allows us to write this expansion in a really compact with with a summation notation, like this:

$$\cos(x) = \sum_{k=0}^{\infty} C_k x^k \tag{25.4}$$

The next trick is to find out what the particular values of the coefficients are. We have a nice notation that tells us what an infinite polynomial will look like. However, the right-hand side could be *any* function. The (infinite) list of coefficients will convert that polynomial into the specific one for the cosine function.

Notice that, so far, we haven't actually done anything to *solve* the problem of finding the polynomial expansion yet, but we have now started thinking about the problem in different ways, and writing it down in different ways, which is usually the first step to solving difficult problems.

Next, we need to find a way to figure out what each coefficient is. You might be thinking, "there's an infinite number of them, that will take foverever! Literally!" However, sometimes, if you are willing to *begin* an impossible journey, you will find that it isn't as impossible as it seems.

25.5 Finding C_0

Now, before we dive headfirst into finding an entire infinite list of coefficients for our infinite polynomial, let's see if we can find a way to determine a *single* coefficient. Just one.

Let's take a look at our polynomial from Equation 25.3:

$$\cos(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \ldots + C_\infty x^\infty$$

Do you notice anything different about that first term? Technically, it is $C_0 x^0$, but, since x^0 reduces to 1, C_0 does not get multiplied by x or any version of x at all. In fact, it is the *only* coefficient in the *entire* polynomial that doesn't get multiplied by x. Can you think of an x value we could try to use which would allow us to ignore all of the other polynomial coefficients?

What would happen if we tried to find the value of the series at $x = 0$? In that case, any term involving x would be set to zero, would it not? Therefore, it would reduce the formula to the following:

$$\begin{aligned}
\cos(x) &= C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \ldots + C_\infty x^\infty \\
\cos(0) &= C_0 + C_1 \cdot 0 + C_2 \cdot 0^2 + C_3 \cdot 0^3 + C_4 \cdot 0^4 + \ldots + C_\infty \cdot 0^\infty \\
\cos(0) &= C_0 + 0 + 0 + 0 + 0 + \ldots + 0 \\
\cos(0) &= C_0
\end{aligned}$$

So, by setting $x = 0$, the entire polynomial drops off except the first coefficient! Now, if you remember your trigonometry, you should have memorized that the cosine of 0 is 1. Therefore, we can then say:

$$\cos(0) = C_0 = 1$$

This gives us the first coefficient for our polynomial!

We still have an infinte number of them left to do, but at least we have a start. By checking the value of the function at $x = 0$ we have determined what the coefficient C_0 is in the polynomial expansion.

We can do this for any function for which we know its value at $x = 0$. For instance, take the polynomial expansion of the function $\sin(x)$. We can find its first coefficient in the same way:

$$
\begin{aligned}
\sin(x) &= C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots + C_\infty x^\infty \\
\sin(0) &= C_0 + C_1 \cdot 0 + C_2 \cdot 0^2 + C_3 \cdot 0^3 + C_4 \cdot 0^4 + \ldots + C_\infty \cdot 0^\infty \\
\sin(0) &= C_0 + 0 + 0 + 0 + 0 + \ldots + 0 \\
\sin(0) &= C_0 \\
\sin(0) &= 0 \\
C_0 &= 0
\end{aligned}
$$

Now, we don't have to write all of that out every time, we can just remember the fact that the value of the function at $x = 0$ will be the value of the first coefficient. So, for instance, if we have the function e^x, we can just substitute in a zero for x to get the first coefficient of its expansion. Since $e^0 = 1$, for the polynomial expansion of e^x, C_0 is 1.

25.6 Finding the Next Values

Okay, bravo. We found the first term of the expansion of $\cos(x)$. What about the rest? There's only infinitely many of them left to go.

Well, let's take a look at that expansion again:

$$\cos(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots + C_\infty x^\infty$$

Remember that, in an equation, we can do anything we want to the equation, as long as we do the same thing to both sides. So, let's take the derivative of both sides, shall we?

On the left side, we have $\cos(x)$. The derivative of $\cos(x)$ is $-\sin(x)$. On the right-hand side, C_0 drops off, because it is just a constant. Then, on the next term, the derivative of C_1x is just C_1, giving us another equation with a single constant term! So, we can determine that:

$$
\begin{aligned}
\cos(x) &= C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots \\
D_x(\cos(x)) &= D_x(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots) \\
-\sin(x) &= 0 + C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + \ldots
\end{aligned}
$$

Now, if we want to know the value of C_1, we can just substitute 0 in for x on our new function, $-\sin(x)$:

$$
\begin{aligned}
-\sin(x) &= 0 + C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + \ldots \\
-\sin(0) &= 0 + C_1 + 0 + 0 + 0 + \ldots \\
-\sin(0) &= C_1 \\
-\sin(0) &= 0 \\
C_1 &= 0
\end{aligned}
$$

Now we know two coefficients! C_0 is 1 and C_1 is 0.

Notice that, since we are dealing with a polynomial, each time we take a derivative, it reduces the power of the x value for the coefficient by 1. Therefore, we should be able to find the nth coefficient using the nth derivative. Below is the continuation of this process through C_4, where we get the next derivative, and then find the value of that function at zero.

$$
\begin{aligned}
\cos(x) &= C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots \\
\cos(0) &= C_0 && = 1 \\
D(\cos(x)) &= D(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots) \\
-\sin(x) &= 0 + C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + \ldots \\
-\sin(0) &= C_1 && = 0 \\
D(-\sin(x)) &= D(0 + C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + \ldots) \\
-\cos(x) &= 0 + 0 + 2C_2 + 6C_3x + 12C_4x^2 + \ldots \\
-\cos(0) &= 2C_2 && = -1 \\
D(-\cos(x)) &= D(0 + 0 + 2C_2 + 6C_3x + 12C_4x^2 + \ldots) \\
\sin(x) &= 0 + 0 + 0 + 3C_3 + 24C_4x + \ldots \\
\sin(0) &= 3C_3 && = 0 \\
D(\sin(x)) &= D(0 + 0 + 0 + 3C_3 + 24C_4x + \ldots) \\
\cos(x) &= 0 + 0 + 0 + 0 + 24C_4 + \ldots \\
\cos(0) &= 24C_4 && = 1
\end{aligned}
$$

Now, if you paid attention, something interesting started happening with coefficient C_2. Notice that we did not find the value of C_2, but rather $2C_2$. $2C_2$ is -1, so that means that C_2 by itself is $-\frac{1}{2}$.

Why do the coefficients starting with C_2 and going further all have factors in front of them, and is there a pattern to them? The reason they have factors in front of them is that they all started out with powers higher than one. When a derivative happened to the term, the power *came out in front*, and then the term dropped to the next power. This continued on until the power reaches zero.

This means that the coefficient will wind up being multiplied by *every number from the power of x it started with down to one*. Another name for this is the factorial! In other words, the multiplier in front of the coefficient will by $n!$, where n is the nth derivative we have taken (the function itself is counted as the 0th derivative, and the factorial of 0 is 1).

The next thing to notice is that the derivatives are going in a circle. cos goes to *sin* goes to − cos goes to sin which goes back to cos and the process repeats. Forever.

Therefore, the result of each step (the nth derivative applied to an x value of zero) is cycling between 1, 0, −1, 0, over and over again. That means that we can write out the polynomial for the cosine function as:

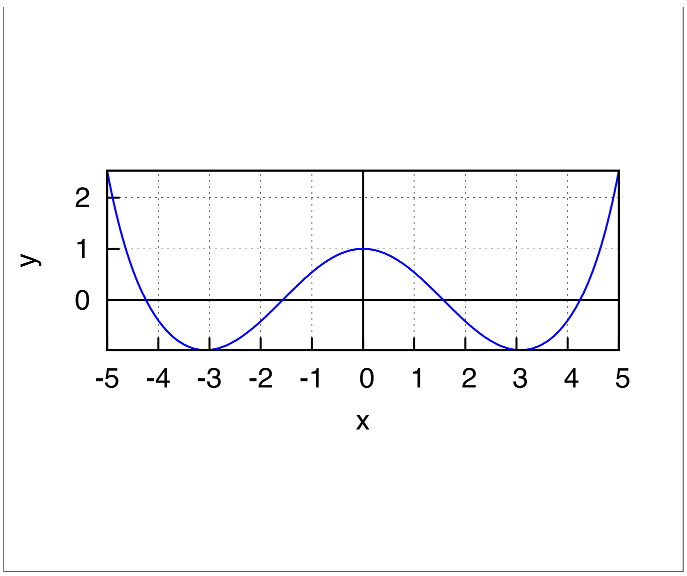
$$\cos(x) = \frac{1}{0!} + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \frac{-1}{6!}x^6 + \frac{0}{7!}x^7 + \frac{1}{8!}x^8 + \dots \tag{25.5}$$

All you have to do to write out the function is to keep cycling the numerator through the sequence 1, 0, −1, 0 and moving the denominator through the factorials, and finally keep the powers of x increasing.

If you do this forever, you will get the full series!

25.7 Using Our Series

Now, even though it takes an infinite number of terms to model the cosine function precisely, even with the few terms we have written out so far, it models it pretty well between −3 and 3:



When it is near 0, the graph really looks and behaves a lot like the cosine function. Tricks like this allow us to write computer programs that get very close to the true values of functions like sine and cosine even though they aren't really native to computers. The programmer can use methods like this to build a program that he can implement on the computer to make up for the fact that there is no real cosine function available.

Let's calculate an example value and see how close we come. The cosine of $\frac{\pi}{4}$ is known to be $\frac{\sqrt{2}}{2}$, which is about 0.707. Let's see how close our first few coefficients get us to. In order to use them, we'll need to expand π out to a number, so we will use 3.14159. Since we are trying to find the cosine of $\frac{\pi}{4}$, we will use a close numeric approximation: $\frac{3.14159}{4} \approx 0.7853975$. So we will be looking for $\cos(0.7853975)$. Substituting it into our polynomial expansion, we find:

$$\begin{aligned}
\cos(x) &\approx \frac{1}{0!} + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \frac{-1}{6!}x^6 + \frac{0}{7!}x^7 + \frac{1}{8!}x^8 \\
\cos(0.7853975) &\approx \frac{1}{0!} + 0 + \frac{-1}{2!}0.7853975^2 + 0 + \frac{1}{4!}0.7853975^4 + 0 + \frac{-1}{6!}0.7853975^6 + 0 + \frac{1}{8!}0.7853975^8 \\
&\approx 0.70710727 \qquad \approx 0.707
\end{aligned}$$

The actual answer has a very large number of decimal points, but we can see that our polynomial is approximately correct for known values of the cosine function.

25.8 Making a General Formula

So, while we have an idea for a formula for the cosine function *in our minds*, we haven't quite yet gotten to the place where we can write it down. Before we get to the precise formula for the cosine function, let's think about a general formula for doing what we have done in this chapter.

We took an arbitrary function, which we can call $f(x)$, and we turned it into a polynomial. Something like this:

$$f(x) = C_0x^0 + C_1x^1 + C_2x^2 + C_3x^3 + C_4x^4 + \ldots + C_\infty x^\infty \tag{25.6}$$

Now, I put in x^0 and x^1 explicitly in this formula, because it will make the next step more understandable. Notice that the entirety of the formula is simply a sum. Well, we have a notation for summation, namely $\sum$. Likewise, each of the terms in our equation follows a very precise formula, so we can rewrite our general formula as:

$$f(x) = \sum_{n=0}^{\infty} C_n x^n \tag{25.7}$$

This sort of a formula takes a lot of the guesswork out of things, because we don't have those little dots (...) which force people to use their imaginations to tell what the next thing is. With this notation, the notation actually tells you how the series progresses.

However, we have also figured out in previous sections what the formula is for C_n. It is the nth derivative of the function evaluated at 0, divided by $n!$. So, we can use that information as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \tag{25.8}$$

In this formula, $f^n(0)$ refers to the nth derivative of f evaluated at 0 (*not* $f(0)$ raised to the nth power), where the 0th derivative of f is the function f itself. This formula makes what is called a Maclaurin series, and works for a large variety of functions—namely smooth, differentiable functions where every derivative is defined (and calculatable!) at zero.

25.9 Finding a Specific Formula

Now, the situation is sometimes more complicated if you want to find a specific summation formula for a specific function. For instance, we *can* write our cosine function with the following formula:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{\cos^n(0)}{n!} x^n$$

Again, $\cos^n(0)$ refers *not* to $\cos(0)$ raised to a power, but to the nth derivative of cos evaluated at zero. Now, we know from experimenting with this in previous sections that this cycles through 1, 0, −1, 0 over and over again. However, it would be nice if we could write this fact into our formula.

When things cycle, I like to use −1 raised to a power to provide cycling. -1^1 is −1, -1^2 is 1, then -1^3 is −1 again. So we can use −1 to provide the ability to cycle.

Now, interestingly, our function has *two* cycles. The first cycle is back-and-forth to zero each time, and the second cycle is going from positive to negative and back. Therefore, we need to develop a formula as a function of n to reflect what any particular coefficient of our function will be.

So, to cycle back-and-forth between zero, we can use the formula $\frac{1+(-1)^n}{2}$. When $n = 0$, this will be $\frac{1+(-1)^0}{2} = \frac{1+1}{2} = 1$. When $n = 1$, this will be $\frac{1+(-1)^1}{2} = \frac{1-1}{2} = 0$. When $n = 3$, this will be $\frac{1+(-1)^2}{2} = \frac{1+1}{2} = 1$. It will continue to cycle back and forth, giving us a multiplier that will zero out the even terms.

The second cycle is back and forth between positive and negative. If we were to cycle *every* time, we would just raise −1 to the nth power. However, because the odd terms will be zeroed out, we need to raise it to the $\frac{n}{2}$ power like this: $-1^{\frac{n}{2}}$. Therefore,

- When $n = 0$, this will be $-1^0 = 1$.
- When $n = 1$, the value will be ignored because the other term will be zero.
- When $n = 2$, this will be $-1^1 = -1$.

So, putting these together, you can get the following formula:

$$\cos(x) = \sum_{n=0}^{\infty} \left((-1)^{\frac{n}{2}}\right) \left(\frac{\frac{1+(-1)^n}{2}}{n!}\right) x^n \tag{25.9}$$

This allows us to specify any arbitrary term, and get the coefficient at that term.

Now, what I like about Equation 25.9 is that it tracks very closely with the Maclaurin expansion we have worked on in this chapter. It is a bit convoluted, but we have made the direct connection between each part of the formula and where we obtained it from.

However, there is a much more popular version of this which has the same result, but it looks a bit cleaner.

$$\cos(x) = \sum_{n=0}^{\infty} ((-1)^n) \left(\frac{1}{2n!}\right) x^{2n} \tag{25.10}$$

The problem with this formulation is that it is harder to see the direct connection between each piece of the formula and the general formula for a Maclaurin series (i.e., Equation 25.8). What this formula does is that, rather than including the terms that go to zero, it simply rewrites the formula so that it only produces nonzero terms. That is why the factorial is for $2n$—it is *only* doing the even ns from the previous formula ($2n$ will always yield an even result).

25.10 Applying the Maclaurin Series

Now that we have learned the basics of how to make a Maclaurin series using the cosine function, let's apply it to a different function—e^x. What we want is a polynomial expansion of e^x. Using Equation 25.8, we can start by writing this as:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

where $f(x)$ is e^x, and $f^n(0)$ is the nth derivative of e^x at zero. But wait—*all* derivatives of e^x are just e^x. And, e^x *at zero* will just be e^0. Anything raised to the 0th power is just 1. Therefore, we can just replace $f^n(0)$ with 1. This gives us the equation:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{25.11}$$

Interestingly, this also gives us an easy way to calculate e itself. Since e is just e^1, we can substitute 1 in for x in this equation, which will give us:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} 1^n = \sum_{n=0}^{\infty} \frac{1}{n!}$$

We can estimate e to any degree of precision we want by simply taking a certain number of terms from this expansion. If we want five terms of this expansion, we will get:

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} \approx 2.708$$

We can get closer by doing seven terms:

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} = \frac{1957}{720} \approx 2.718$$

The more terms we add, the closer and closer to the true value for e that we will get.

25.11 Another Way of Looking at the Maclaurin Series

While the Maclaurin series is a nice trick, some people have trouble wrapping their minds around what is really happening here.

Let's say that we go outside and look at the thermometer. The thermometer say it is 67°F outside. Now, it's good to know what the temperature is *now*, but what is the likely temperature in two hours? Let's create a function called $temp(x)$ that returns the temperature x hours from now. So, the current temperature is $temp(0) = 67$.

Well, assuming that we don't know anything about how temperature works, or what the weather is like in our area, the best estimate that we can make for the temperature in 2 hours is to simply say that it is going to be the same temperature as it is now.

Therefore, our first approximation for the full temperature function is going to just be $temp(0)$. In other words, we will guess that:

$$temp(x) = 67$$

However, let's say that someone tells us that the temperature is *changing* 3 degrees per hour. That means that the *derivative* of the temperature right now is 3. In other words, $temp'(0) = 3$.

Therefore, we can create a line containing both pieces of information—the temperature that it is now and how it is changing. The resulting equation will be:

$$temp(x) = 67 + 3x$$

Now, let's say that the same person tells us that not only is the current rate of change of the temperature change 3 degrees per hour, but that rate itself is changing −1 degree per hour per hour. That means that $temp''(0) = -1$. Because it is "per hour per hour", that is the same as "per hour squared." Therefore, it will be multiplied by the square of the current time (x) to balance out units. However, its total impact will be less, because it takes a while for it to take effect, so we will divide this number by two.[2]

So, our new equation is:

$$temp(x) = 67 + 3x + \frac{-1x^2}{2}$$

As you can see, we can write this just like our Maclaurin series expansion:

$$temp(x) = \frac{temp(0)}{0!} + \frac{temp'(0)}{1!}x + \frac{temp''(0)}{2!}x^2$$

The more we know about how the temperature is changing through its various derivatives, the better we will be able to create an equation to model the temperature.

Each derivative will give us more information about how the temperature is changing, which we can continue to add into our model and expand. A perfect model of temperature change would likely include an infinite number of terms, but by adding more and more terms, we can get closer and closer to the true value of the function using a polynomial model.

25.12 A General Overview of Solving Impossible Problems

We spent a lot of time in the details of finding the polynomial expansion of functions. However, I want to take a moment and go back and review the general steps we took for finding the solution of an impossible problem.

[2] The technical reason you would divide by two is that the total impact of this derivative is obtained by integrating it twice. The first integral would be $-1x$, and the second integral is $\frac{-1x^2}{2}$. Each additional term would have to be integrated yet another time to reach its total impact. This is what causes the bottom term to be consecutive factorials—each time you integrate, you divide by the next number. By the time you get to the nth term, you have to integrate n times to find the total impact, which will cycle through dividing by all of the integers from 1 to n.

1. We started by naming the problem. We established what the impossible problem was, and what it was that made the problem impossible.
2. Next, we identified the assumptions that made the problem impossible.
3. Then, we removed the assumptions that made the problem impossible to see if the problem is now solvable.

Now, this didn't solve our problem. In fact, while it removed the "impossibility" part of the problem, it introduced a new set of problems. Many times, on the road to solving a problem, you may find yourself lost. If you look back to Equation 25.1, we were pretty lost ourselves. Here are some things that you can do to get yourself out of being lost:

1. Play with how the problem is written. In our case, we had to rewrite the problem numerous times before we had a version that we could easily work with.
2. Identify what it is that is making the problem difficult. In our case, we identified two problems—that the subscripts were going to infinity from different directions, and that we were having to write out both the finite and infinite cases. These were both solved by rewriting the problem in a new way.
3. Find the solution to a subset of the problem. Our biggest breakthrough happened when we found C_0. The first key insight was that evaluating the function at $x = 0$ isolated C_0 among the infinite sea of coefficients. We would never have solved the problem by trying to solve the whole thing at once. By focusing on just one super-tiny subset of the problem, we were able to get started.
4. Find a mechanism to build success upon success. In our case, we figured out that taking the derivative of the polynomial caused the next coefficient in line to reduce its power to zero so it could be isolated. This allows us to find the "next" thing in line.
5. Identify patterns in the result. Building success upon success enough times allowed us to identify the pattern which was emerging. Once we could see the pattern clearly, we didn't need to solve for all infinity terms. We could write a formula to do it for us!

Anytime you are stuck in a mathematics (or other!) issue, I hope you come back here are remember that, even if the problem is indeed impossible, you have the tools to solve it.

Review

In this chapter, we learned:

- Polynomials can be used to approximate most other smooth functions given a sufficient (possibly inifinite) number of terms.
- Computers can compute approximate values of many smooth functions by using truncated versions of these infinite polynomials.
- For most smooth functions, increasing the number of terms in the polynomials yields better accuracy for the result.
- For infinite series of polynomial terms, it is usually beneficial to arrange them in order from the *lowest* power of x (starting with x^0, the constant) to the highest power of x (i.e., x^∞).
- The first (constant) term of the polynomial can be found by finding the value of the function at 0.

- Each term can be isolated by taking the nth derivative of the original function, finding the value at zero, and dividing by $n!$.
- The summation equation for this is given in Equation 25.8, and is known as the Maclaurin series for a function.
- When writing formulas for the Maclaurin series of a function, finding a formula for the constant is sometimes difficult.
- When writing formulas for the Maclaurin series of a function, creating cycles for the constant can often be done by raising -1 to a power related to n.
- when someone says something is "impossible," sometimes even if they are right, it is often only impossible because of certain assumptions they have made. There may be other assumptions that cause different conclusions.
- The best way to solve an "unsolvable" problem is to find the assumptions that people might not even recognizing that they are making.
- Removing assumptions can make impossible things possible.

Exercises

1. Use the process in the book in Sections 25.5 and 25.6 to find the first six coefficients for the polynomial expansion of $\sin(x)$.
2. Use the process in the book in Sections 25.5 and 25.6 to find the first six coefficients for the polynomial expansion of 2^x.
3. Using the previous two problems, find an approximate value for $\sin(1.1)$ and $2^{1.1}$. Compare these answers to what your calculator yields (don't forget to use radians!).
4. Try to come up with a formula for $\sin(x)$ that is similar in style to Equation 25.9.
5. Estimate the value of e using the first ten terms of the expansion of e^x. Round off to eight decimal places. Look up the value of e in a book or on the web. How many decimal places were you accurate to?
6. Use Equation 25.8 to find the $n = 8$ term for the expansion of e^x.
7. Using Equation 25.8, find the first four terms of a polynomial representation of $x\,e^x$.
8. Use the previous question to write a formula for every term of the expansion of $x\,e^x$.
9. Write out the expansion of $\cos(x)$ to ten terms (including the zeroed out terms) using Equation 25.9.
10. What is the term for $n = 513$ (i.e., the 514th term) of the expansion of $\cos(x)$ according to Equation 25.9? If there are any factorials in the result, leave them as factorials.
11. What is the term for $n = 8214$ (i.e., the 8215th term) of the expansion of $\cos(x)$ according to Equation 25.9? If there are any factorials in the result, leave them as factorials.