Communications of the Blyth Institute

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Blyth Institute Press

CBI Communications of the Blyth Institute

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About This Journal

1 The Purpose of the Journal

Communications of the Blyth Institute was founded due to needs both within The Blyth Institute as well as needs in the wider research community. As The Blyth Institute has grown, it has become more difficult to share and disseminate research and ideas for research within our own community. Mailing lists are ephemeral, blogs are trite and difficult to cite in later research, and forums tend to be worse on both accounts. Therefore, this journal was established to provide a means of communicating ideas that is more formal than conversation, but also more expeditious than other journals.

Likewise, in the wider research community, ideas often need time as well as participation to grow. That is, the first iteration of an idea, even of a great idea, is often fraught with problems that need to be ironed out. Most journals quite reasonably want to have ideas more fully formed before publication. However, sharing ideas informally prior to publication increases the risk for researchers of others taking credit for their ideas.

Therefore, *Communications of the Blyth Institute* fills a role for creating a space for the formal publication and peer review of inchoate ideas, both inside and outside of The Blyth Institute.

2 Paper Submission Policies

2.1 Submitting to the Journal

Because the primary goal of *Communications* is to serve the membership of The Blyth Institute, all submissions must either be by a Blyth Institute member or be sponsored by a Blyth Institute member. "Sponsorship" in this case merely means that the Blyth Institute member believes that the paper is worth considering by members of The Blyth Institute. The sponsoring member does not have to have any official association with the paper. If someone who is not a member of The Blyth Institute wishes to submit a paper for consideration, all that is needed is for them to submit the paper to someone who is a Blyth Institute member and ask that they sponsor it for you. The sponsoring member will then submit it on your behalf.

Members of The Blyth Institute may submit their ideas to communications@blythinstitute.org.

2.2 Submission Review Process

Communications employs a prompt review process for formal papers submitted to the journal. The editor will appoint one or more reviewers to review the paper. Reviewers will review the paper to verify that the paper fulfills the following qualities:

- The idea is original (unless explicitly marked as a review paper).
- The idea is of interest to either Blyth Institute members or the wider research community.
- The author makes reasonably clear distinctions between assumption, fact, inference, proposal, opinion, and conjecture.[1]
- The author presents reasonable data and evidence for their conclusions.

Additionally, reviewers should provide commentary on the ideas themselves. Reviewers may weigh in as to whether the paper should be published, though ultimate publication authority lies with the editor.

Authors will then be given a chance to revise their paper and respond to the comments and cricitisms of the reviewers. The editor will then make a final determination for publication.

[1] Because this journal focuses on more inchoate ideas, we want readers to be clear how well-founded each part of the author's proposal is. The goal isn't to make an overly-wordy paper, but simply to make sure readers are appropriately informed as to the levels of confidence that should be attached to various ideas presented in the paper.

2.3 Scope of the Journal

The scope of the journal is not strictly limited. The Blyth Institute is a loose consortium of researchers scattered throughout the world, with many diverse interests. The main focus of The Blyth Institute has been in non-reductionist perspectives on biology and cognition. While this will likely remain the focus of the journal, any topic that a Blyth Institute member thinks that other Blyth Institute members will benefit from can be included. *Communications* accepts papers in a wide variety of fields including most sciences, mathematics, and philosophy.

The editors have the final say for inclusion and exclusion for the journal.

2.4 Formatting Papers for Submission

Ideally, papers should be submitted in LaTeX format using an "article" document class. Because the journal is presently run entirely by volunteers, we request that papers utilize a minimum of LaTeX trickery, focusing instead on basic LaTeX features to simplify inclusion in the journal.

If you are unfamiliar with LaTeX, you may submit your file as a Word document (.doc or .docx). Please keep formatting to a minimum, and submit all figures and tables as *separate* files.

For smaller submissions such as news items, letters, and notes, feel free to simply email them directly as text.

3 Other Journal Content

3.1 Student Papers

The Blyth Institute has always recognized the importance of enabling the next generation of researchers. As such, we welcome contributions from students. The Blyth Institute recognizes that students do not always research and write on the same level as more established researchers, as their breadth of experience and knowledge is not the same.

Therefore, The Blyth Institue will also allow papers from students that undergo a lighter level of review, and for which our standards are relaxed. These papers will be marked as "Student Papers." Any student is free to make regular submissions as well. Readers should be aware, however, that papers marked as student papers will have relaxed standards applied.

3.2 Letters and Notes

Communications will also publish letters and notes sent to the editor. These letters and notes can be for a variety of purposes, including but not limited to (a) responding to a previously published *Communications* paper, (b) responding to publications elsewhere, (c) responding to news events or cultural aspects of science, and (d) communicating short ideas that have not yet been developed into paper-length submissions. Letters and notes are primarily reviewed by the editors, but the editors, at their discretion, can request additional review from other sources.

3.3 Tutorials and Reviews of Fundamentals

In addition to typical reviews covering the latest results in a field, *Communications* also publishes tutorials and reviews of the fundamentals of a field, aimed at providing experts in other fields information that they may need for cross-disciplinary work. Tutorials focus on the performing of a task while a fundamentals review focuses more on concepts.

3.4 Book Reviews

Communications encourages the submission of book reviews, especially by newer contributors. Book reviews are an excellent way for new researchers to both gain knowledge in their field as well as publication experience. Book reviews for *Communications* should include both summaries and critical engagement with the material. The books should be scholarly works which would be of interest to Blyth Institute members.

3.5 News Items

Communications also supplies a news section. This section will include news about The Blyth Institute itself, as well as anything that Blyth Institute members find interesting and worth sharing with other members. If you are a member of The Blyth Institute, notices of peer-reviewed papers published elsewhere will appear within the news section.

From the Editors

To start our third year at *Communications of the Blyth Institute* (CBI), we want to say a word of thanks to our contributors and our readers. Our growing contributor and reader base has helped make CBI successful.

Looking at our download statistics, our most downloaded paper was the paper featured in our first issue, Albert de Roos (2018) "A Proposed Framework for Cellular Evolution," while the paper with the most overall interest was Steve Dilley and Nicholas Tafacory (2019) "Damned if You Do and Damned if You Don't." We have had many papers receive over a thousand downloads from the website. The paper with the most citations so far is Jonathan Bartlett and Eric Holloway (2019) "Generalized Information: A Straightforward Method for Judging Machine Learning Models." That paper has been cited in Ph.D theses, AI books, and even some biology journals (*BMC Biology* and *BIO-Complexity*).

One of the goals of CBI is to be an incubator for new ideas. Several authors have already used CBI in such a way. Several CBI papers by Eric Holloway became the foundation for a paper he co-authored with David Nemati, "Expected Algorithmic Specified Complexity." Hunter's "The Random Design Argument" was a foundation for his later paper "On the Influence of Religious Assumptions in Statistical Methods Used in Science."

The letters section of CBI is intended as a discussion forum to help researchers develop new ideas and comment on each others work. We have seen that happen with the discussions around Bartlett, Gaastra, and Nemati (2020) "Hyperreal Numbers for Infinite Divergent Series," for which we have had letters from multiple scholars noting how that methodology can be used for analyzing various problems.

Though we are an admittedly small journal, we want to thank the contributors and readers for giving us an outsized success.

Sincerely,

—The Editors

When is Explanation Transitive? A Methodological Note

Sam S Rakover and Baruch Cahlon

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Abstract

The article considers the following explanatory-transitivity question: assume that A explains B and B explains C: does A also explain C? In the present paper the term explanation signifies causal explanation. The discussion of this question arrives at the answer that a necessary condition for explanatory-transitivity can be proposed. Accordingly, if B explains observation O (bEo) and A explains B (aEb), then A also explains O (aEo), when: if the same E is not preserved in the three expressions (bEo, aEb, aEo) then the transitivity of E will not be preserved. This answer is supported by an analysis of a large number of examples. The article also analyzes the relations among explanation, reduction and transitivity.
Keywords: *methodology, explanation, transitivity*

1 Introduction

The question discussed by this article is this: assuming that A explains B and B explains C, does A also explain C? In other words, does the relation of explanation E uphold transitivity? This is a very difficult question to answer, since there are many different models of explanation (for review see Psillos, 2007; Rakover, 2018; Salmon, 1990; Woodward, 2011). Hence, a qualification is needed. While an explanatory model based on deduction (such as the Hempel's D-N model) does realize the transitivity requirement (if A is deduced from T and T is deduced from T^*, then A is also deduced from T^*), other models do not always do so. A major purpose of the present paper is to show that an explanatory model based on the concept of causality (e.g., Salmon's 1984 causal-mechanical model which is based on the concept of causal process) meets the transitivity requirement under special condition as described below. (Note that hereinafter the term "explanation" signifies causal explanation.)[1]

To answer our opening question, we shall first discuss briefly the concept of transitivity, and then, based on this discussion, we shall deal with the question considered by the article. While the transitive relation is dealt with in the theoretical realm, the explanation relation is dealt with in the empirical realm. When one deals with empirical relations it is not always clear whether one is dealing with unidimensional or multi-dimensional relations, and whether the same relation holds in all comparisons. Hence, one may view our question in the following way: does the theoretical concept of transitivity hold in the empirical domain? To answer, we have to check whether transitivity holds in reality, which can be determined by means of the process of measurement and realization. This discussion is methodological in the main, because E (explain) in science applies to the connection between theories and empirical observations, that is, between the *explanans* (the explainer) and the *explanandum* (the to-be-explained).

The theoretical requirement for a transitive relation is: if x relates to y by the relation R (expressed by xRy) and y relates to z by the same relation R (yRz), then x relates to z by the same relation R (xRz). In other words, for all events x, y, z, when the three expressions xRy, yRz and xRz obtain, then R displays the property of transitivity. As mentioned above, this requirement is accepted in theory. However, its application to reality may cause some problems especially when the relation among x, y, z, is not unidimensional, for example, stronger than, prettier than, smarter than. As we shall see, the application of the theoretical term transitivity to the empirical realm scientific explanation is complex. We attempt to show that when one applies the term transitivity to reality, it fulfills the requirements of a necessary condition (that the same relation R appears in the three expressions xRy, yRz and xRy) but not those of a sufficient condition.

Given the above clarification, we shall now analyze several examples while emphasizing the application of this theoretical term to the empirical realm.

Let us study the relation "higher than" (H): H is transitive

[1] The concept of causality is very complex and is under continuous debate in the philosophical literature. Hence, its discussion here is beyond the purpose of this note. We shall delineate the concept in the following way: it is considered as an answer to, explanation of, the question 'why'. It proposes that event B (the effect, the phenomenon under investigation) depends on a preceding event A (the cause), when both events are close in time and space and show invariable correlation.

when x is higher than y, xHy, and yHz, then xHz for every x, y, z.[2] If one does not assume axiomatically that the same H appears in the three expressions among x, y, z and displays transitivity, the following methodological question arises in the empirical realm: how can one be certain that H is indeed transitive? The answer is as follows: we ensure this by using the same procedure of measuring the height of x, of y and of z. The results of the measurement show that indeed it is the case that if x is higher than y and y is higher than z, then x is also higher than z, i.e., the transitivity relation applies to the present case. Similar results are obtained for the relation "heavier than" (measurement is taken on scales) and "older than" (measurement is by means of time). However, as we shall see below, there are several cases in the real world that do not uphold transitivity.

As may be seen from the above examples, the same R (relation) is ascertained in the three comparisons (xRy, yRz, xRz) by means of a valid and reliable measurement, namely a process that measures only what it is supposed to measure and gives the same measurement at different times. When R is not the same, transitivity is not preserved because the relations in the three expressions deal with different matters and different subjects. For example, take the relation "prettier than": x is prettier than y and y is prettier than z—but in many cases people will judge z prettier than x! Why? Because the relation "prettier than" is not identical in all the three expressions. For example, the judgment is that x is prettier than y because of her hair, and y also is prettier than z because of her hair, but z is prettier than x because of her wonderful eyes. And another example: assuming that x is taller than y and y *is prettier* than z, we shall not be surprised that z is taller than x! For example, x's height is 1.80 m and y's is 1.70 m, but z's height is 1.90 m; z is plain looking while y is beautiful. Based on these examples, it may be suggested that the same R must appear in the three expressions, because if it changes in one of them, then as we have illustrated above transitivity will not be preserved. Here is an example taken from the world of sport, which illustrates that causal transitivity is not always preserved. David defeated Max in a wrestling contest; Max defeated Dan in a boxing contest; however, Dan defeated David in a wrestling contest. How is this possible? One interpretation is as follows: David is a mediocre wrestler; Max is an excellent boxer and Dan is an outstanding wrestler. Hence, we are dealing here with two different relations: one is connected to wresting and the other to boxing and transitivity is not preserved.

[2] In many cases we transform properties and actions into relations. For example, we transform the property "height" into the relation "higher than," and the action "preference" into the relation "prefer x more than y."

1.1 Preservation of R and transitivity

The question arising here is this: is the requirement of the existence of the same relation in the three expressions a sufficient condition or a necessary condition for transitivity? In our view the answer is that the requirement is necessary because it can be shown that transitivity is not always preserved even though the same R appears in the three expressions. We may examine the "threshold relation". Assume that a certain threshold relation holds when Threshold Relation (T) obtains. This relation holds if the multiplication of two of its three values (x, y, z) is larger than or equal to 0.5 ($i \cdot j \geq 0.5$). (The values arise from the following dimension: 0.0…0.5…1.0.) Assume the following values: x=0.7, y=1.0, z=0.6. And now, may one propose that because the relation is held in the following two expressions: xTy =0.7, where T indicates the multiplication function, and yTz =0.6, the relation will hold in the interaction xTz? The answer is negative, because $xTz = (0.7)(0.6) = 0.42 < 0.5$, therefore transitivity is not preserved.

In light of the present discussion, the following proposal may be raised: a necessary condition for transitivity to be preserved is that the same R be preserved in the three expressions: xRy, yRz, and xRz, for every x, y, z. That is, if relation R is not maintained in the three expressions transitivity is not preserved.

1.2 Transitivity and dimensionality

Based on the foregoing analysis it seems possible to make two proposals concerning the transitivity that appears in one-dimensional phenomena (e.g. a straight line in geometry) and in multi-dimensional phenomena (e.g., a plane or a cube in geometry).

- *Proposal (a)*: When R is applied to a one-dimensional phenomenon (height, weight, time) transitivity will be preserved. To illustrate this proposal let us examine once again the relation H. Given xHy and yHz, is it possible that xHz will not obtain? The answer is negative for this reason: given that H can be expressed as follows: xHy is stated by (x-y)>0, yHz by (y-z)>0 and xHz by (x-z)>0, then [(x-y)+(y-z)=(x-z)]. That is, if xHy and yHz, then xHz.

  This support notwithstanding, a case can be brought that contradicts proposal (a). It is possible to interpret the threshold relation described above in such a manner that it will contradict proposal (a). According to this interpretation, the threshold relation is a one-dimensional whose values are between 0.0 and 1.0. And because $xTz < 0.5$, although $xTy > 0.5$ and $yTz > 0.5$, transitivity does not occur in this one-dimensional. Therefore, it is not possible to state that one-dimensionality is a sufficient condition for preserving transitivity.

- *Proposal (b)*: Contrary to (a), when R is applied to a multi-dimensional phenomenon transitivity is not necessarily preserved (for instance, see the above example "prettier than"). It is hard to support this suggestion, because many phenomena, such as speed or acceleration, are multi-dimensional but display transitivity. (E.g., if in a racecar competition car A reaches the finishing line faster than car B and car B faster than car C, then car A has finished the race faster than car C. Note that in this illustration even though speed is defined multi-dimensionally by distance and time, the speed itself is expressed as a one-dimensional number.) And in complete contrast, in a large number of phenomena (prettier than, preference for goods, victory in a sports competition such as soccer, etc.) transitivity is not always preserved. (The appendix gives some examples and proofs for the fact that relations applied to multi-dimensional phenomena may sometimes preserve transitivity and sometimes not.)

  Applying the distinction between descriptive and normative models in science to our concerns, we may suggest that the empirical approach to transitivity that we posit here is descriptive. According to Bell, Raiffa, and Tversky (1988), who discuss decision making, a normative model offers a rational way whereby people *must* make decisions; and a descriptive model describes the way people make decisions *in practice*. Most decision-making models that assume that people will behave rationally assume that their decisions will be transitive: if A is preferred to B and B is preferred to C, then A will be preferred to C. It transpires that in many cases people behave in keeping with this assumption.

2 Explanatory-transitivity

On the basis of the discussion so far, it may be suggested that E is transitive if it meets the following requirement, called *explanatory-transitivity*:

> If B explains observation O (bEo) and A explains B (aEb), then A also explains O (aEo), when: (a) if the same E is not preserved in the three expressions (bEo, aEb, aEo) then the transitivity of E will not be preserved; (b) at least one realization of E may be posited (B symbolizes a hypothesis or simple theory and A symbolizes a broader and deeper theory).

The present requirement of explanatory-transitivity acts as a necessary condition because the above discussion showed by several examples that transitivity is not preserved when R is not preserved in the three expressions.

To substantiate this statement, we shall analyze several examples that preserve transitivity and several that do not.

2.1 Example: Free Fall

Galileo's law explains the free fall of bodies; Newton's theory explains Galileo's law; and Newton's theory also explains the free fall of bodies. The fall of bodies may be explained by means of Galileo's law when the D-N model is used (see Hempel, 1965); Galileo's law can be explained by means of Newton's theory when in this case also the D-N model is used (i.e. by means of this model it is possible to explain particular facts and also general regularities. Still, it is worth stressing here that what is derived from Newton's theory is a very good approximation of Galileo's law, and that only in proximity to the earth does gravitation behave as a constant); and Newtonian theory explains the fall of bodies by means of the D-N model. In all these relations it is possible to meet the requirement of explanatory-transitivity by the D-N model when inserted into it are the proper law or theory and empirical conditions, and when what stems from this model corresponds to the empirical observation or to the empirical generalization (Galileo's law).

2.2 Example: Pool Balls

A chain of events on the pool table is offered as an example of causal transitivity (e.g. Halpern, 2016). For example, three balls are on the table. The eight-ball is near the upper right-hand pocket; the seven-ball is approximately in the middle of the table; the (white) cue ball is in the lower part of the table. (a) The cue strikes (S) the cue ball (c) which strikes the seven-ball (7) (cS7); (b) the seven-ball strikes the eight-ball (8) and pockets it (7S8). Is transitivity preserved here? That is, is cS8 realizable? The answer is affirmative. Namely, the requirement that the cue ball, which has been struck by the cue, directly strike and pocket the eight-ball is realizable. Furthermore, pocketing the eight-ball in the corner pocket is realizable in various colorful ways—some of them entertaining (depending on the pool player's virtuosity).

2.3 Example: The Handgun

This example has the following chain of events. (a) The trigger of the handgun is squeezed and activates the hammer which strikes the cap. (b) The strike on the cap explodes the mercury fulminate inside, which ignites the gunpowder in the bullet's cartridge. (c) The gases created in the ignited cartridge expel the bullet from the gun's barrel over an effective range of about 25 meters. These events may be expressed by means of causal explanations: aE_1b (pressure on the trigger explodes the gunpowder in the cartridge); bE_2c (the gases formed as a result of the explosion in the cartridge propel the bullet from the gun's barrel over an effective range of about 25 meters). That is, the explanations E_1, E_2 are different (mechanical, chemical/physical) and transitivity is not preserved. It is hard to see how pressure on the trigger in itself will propel the bullet an effective distance of about 25 meters.

2.4 Example: A Mother Of

The relation "a mother of" is considered a clear example of absence of transitivity. Why? Because if A is the mother of B, and B is the mother of C, then it is not possible for A to be the mother of C. However, it is possible to conjecture several situations in which the notion of motherhood changes in such a manner that in the new framework with new possible realizations transitivity is preserved. For example, it is possible to expand the concept of motherhood beyond the biological function and to speak of a mother who raises and educates the child. A situation can be imagined where immediately after the birth B died and A raised and educated C all her life from the moment of birth; or a case where B died of a serious illness, but before that her fertilized egg was implanted in A, so that A gave birth to C, and also raised and educated her. In these cases, it seems that realization of the aims of motherhood has been preserved, as well as transitivity.

2.5 Mechanistic vs. Theoretical (M/T) transitivity

The three foregoing examples—"pool ball," "handgun," and "a mother of"—are illustrations of empirical phenomena about which it may be asked *What is the causal mechanism responsible for their occurrence?* In this connection we may raise the question of transitivity: is the first event which activates the mechanism (the strike of the cue; the squeezing of the trigger; A is the mother of B) until the occurrence of the concluding event (pocketing the eight-ball; propulsion of the bullet from the barrel; birth, education, and raising of the infant C) likely in itself to cause the occurrence of the concluding event? By contrast, the illustrations of higher than, heavier than, lasts much longer than, do not deal with occurrences that can be explained by a causal mechanism. There is no causal connection between the height of x, of y and of z in the sense that x influenced the height of y, or the reverse. The transitivity question in the present case stems from the theoretical structure present in the observer's cognitive system. For example, the observer sees x, y, z, grasps through measuring that these three objects differ in height, that x is taller than y, y is taller than z, and finds that x is taller than z. It seems that xHz is unrelated to any mechanism that connects these three objects. (For a similar interpretation see Jones, 2000.)

The difference in the M/T transitivity explanation, namely the distinction between the first three and the last three phenomena, is that for an understanding of the transitivity in the former phenomena an explanation resting on the causal mechanism is suitable, while for the latter phenomena a theoretical, non-causal, explanation is suitable. To highlight this difference, we offer another example, the *square and the circle,* which will substantiate transitivity based on geometrical considerations. We may look at the following relations: (a) before us is a hard surface in which there is a round hole, c_1, of diameter d_1, and a rigid square s of diagonal h. The square will go through the circle when $d_1 > h$; (b) before us is a rigid surface in which is a square hole s, of side s (and diagonal h) and a rigid circle, c_2 of diameter d_2. The circle will go through the square when $s > d_2$; (c) will c_2 go through c_1? The answer is affirmative because ($d_1 > h > s > d_2$) obtains.

3 Discussion

In this section we briefly discuss three matters: first, we examine the main issue of the present paper: can one apply the theoretical concept of transitivity to the domain of scientific explanation of empirical phenomena? Secondly, we examine what the present approach has to say about an example concerning the absence of transitivity in Halpern (2016); thirdly, many researchers have suggested that a phenomenon acquires an explanation when it is possible to reduce it to basic components (e.g. Jones, 2000). Because in science there is an interesting connection between giving an explanation and reduction, we shall try to grasp the connection between the present approach to explanatory-transitivity and scientific reduction.

3.1 Application of the concept of transitivity

Does the application of the transitivity-concept depend on certain internal properties of the events to be explained? For example, the relation "heavier than" is about a certain property of the events discussed. Thus if David is heavier than John and John is heavier than Dan, then David is heavier than Dan. However, the relation "a distance greater than" is not a property of the event (object) under discussion. In this case the relation of transitivity depends on the geometrical structure of the three events under discussion. For example, if the three events are lined up, then transitivity holds. So if the distance between David and John is greater than the distance between John and Dan, then the distance between David and Dan is the greatest. However, if the three events are arranged in a triangle, then in many cases transitivity does not hold. What does this discussion imply for the application of the transitivity-concept to explanation? Suppose one wishes to explain the phenomenon regarding "heavier than." We believe that there will be no problem to propose a theory that explains why David's weight is 90kg, John's weight is 80kg and Dan's weight is 70kg. Given that the same explanation holds for all these three events, then according to the proposal of necessary condition, "explanatory-transitivity," there is no obstacle to applying the transitivity-concept successfully here. However, the situation is a bit more complicated with regard to the "a distance greater than." Suppose that one may construct a geometrical theory explaining why the distance between David and John is greater than the distance between John and Dan. Can this theory explain why the distance between David and Dan is the greatest? The answer is yes. Since the theory is geometrical, it will be able to specify when the answer is yes (transitivity holds, since the three are lined up) and when in many cases the answer is no (transitivity does not hold, since the three are not lined up).

Another question regarding the present topic is the following: can one apply the concept of transitivity to empirical explanation when the transitivity is based on a combination of relations?[3] In many cases the explanation of a phenomenon is based on a combination of different causes and mechanisms. Take for example the handgun example discussed above. Clearly the explanation for the expulsion of the bullet from the gun is complex and based on different mechanisms and theories. Therefore, according to our proposal of "explanatory-transitivity," transitivity is not preserved in this case.

Another example of a combination of relations is the possibility of disjunction. Consider the following relation "faster or smarter": A is faster and smarter than B and B is faster than C, but as it turns out C is the smartest. One may construct a theory for speed of action and another theory for smartness and explain by these theories why A is faster and cleverer than B and why B is faster than C (when on the cleverness scale the two are ranked unequally) and why C is cleverer than A. So a disjunction of relations does not always preserve transitivity. The point to emphasize here is this: in the realm of experimentation one may test several factors to find out how they explain behavior. One may expect to obtain transitivity if the responses (outputs) on the Y-axis are increasing in parallel as a function of the increase in the stimuli (inputs) on the X-axis. Usually, however, one obtains interactions that do not enable transitivity. (For additional complex situations see the above discussion regarding multidimensionality.)

[3] Lange (in press) discusses the combination question in the framework of the conception of transitivity as a principle that helps proposing an argument against Hume's account of natural law. This issue is not under the spotlight of the present paper.

3.2 Transitivity and the "bite of the dog" example

(a) Jim planned to detonate a bomb by pressing a button with his right hand: (b) a dog bit Jim's right hand; (c) Jim pressed the button with his left hand and the bomb went off. According to Halpern's analysis the bomb would have exploded had Jim used his right hand (in the counter-factual case where the dog did not bite that hand) or his left. But pressing with his right hand is not possible when the dog has bitten this hand. Therefore, Halpern holds that causal transitivity has not been preserved. (This example is based on McDermott's 1995 article which critiques Lewis's approach to causality and uses it against causal transitivity.) By contrast, according to the present approach transitivity is maintained because pressure by the right hand represents one possible way of realizing Jim's intention, that is, one possible cause of the explosion. This intention could have been realized by pressing with the left hand also. Actually, Jim's intention of pressing the button could be realized in other ways (e.g. a friend who obeys Jim's will). Given that the explanation relation may be realized in various ways, for this example one may suggest an interpretation where transitivity is maintained.

3.3 Reduction

The reduction of theory B (T_B) to theory A (T_A), which is more basic and extensive than T_B, has generated several approaches, diverse models, around which many debates ensued (see Jones, 2000; Ney, 2016; van Riel and Gulick, 2016). In the present framework we shall discuss briefly the following approaches, which seem most relevant to the problem of explanation and transitivity.

The D-N model

By this model particular events as well as empirical generalizations can be deduced (explained). Taking Galileo's law as an empirical generalization, it is derivable by appeal to Newton's theory. In the present case it can be suggested that the reduction of T_B by T_A can be understood that T_B is explained by

T_A. In our view an important problem in this approach is that the present model does not deal properly with the relation between the terms of T_B and the terms of T_A (e.g., according to Newtonian theory gravitation varies as a function of distance squared between two masses; according to Galileo's law g is constant). The following approach attempts to handle this issue.

Nagel's model

By Nagel's (1961) approach to theoretical reduction, T_B is reduced by T_A if it is possible to deduce, derive T_B from T_A by the use of bridging laws that connect the terms of T_B with the terms of T_A. This approach, which in structure is like Hempel's explanatory model (the D-N model), sparked great criticism on the one hand, but also abundant defense on the other (see Jones, 2000; Ney, 2016; van Riel and Gulick, 2016). Nevertheless, we decided to concentrate on Nagel's model, because it is considered a cornerstone in the field of reduction. Here, the interesting question we consider is this: what is the relation between reduction and explanatory-transitivity? One may argue that if the reduction of T_B by the more basic theory T_A is possible then in this case explanatory-transitivity will also be obtained. That is, if T_B explains O, and if T_B is subject to reduction by T_A, then T_A will also explain O. Against this argument the following reservation may be raised.

As mentioned, explanatory-transitivity rests on the same relation R obtaining for the three expressions, and on it being possible to realize R in various ways, that is, on multiple realizations. But multiple realization is deemed one of the strongest arguments against Nagel's reduction model: it is not possible to propose bridging laws between the terms of T_A and the terms of T_B because these theoretical terms are subject to different realizations. For example, the term referring to a feeling of heat is likely to have different neurophysiological realizations in humans and in diverse animals. That is, while by our approach multiple realizations are the basis of preservation of transitivity, this multiplicity erects a barrier to the reduction process.

Given this, it is reasonable to suggest that there is no identity between explanation and reduction. On various grounds other researchers have reached a similar conclusion, namely understanding reduction as a deductive process does not accord with the development of science (e.g. Jones, 2000; Ney, 2016; van Riel and Gulick, 2016).

To conclude, the present article stresses the following two central points. First, it is possible to propose a necessary condition for explanatory-transitivity. If the scheme of explanation is not preserved in the three expressions, then transitivity will not obtain. This argument is supported by an analysis of several examples. Secondly, the discussion of the connections among explanation, reduction and transitivity showed the following: while multiplicity of realizations is an important component for the necessary condition of explanatory-transitivity, many consider it a barrier to reduction according to the model of Nagel (1961).

	1	2	3
A	10	9	8
B	5	7	6
C	3	7	7

Figure 1: Preservation of transitivity of greater-than

Appendix: Multi-dimensional Relations Sometimes Preserve and Sometimes Do Not Preserve Transitivity

Figure 1 offers a transitivity example for the relation *greater than*, where the row presents three dimensions and the column three events (objects, properties, persons, etc.).

We define the relation *greater than* as follows: $A - B > 0$ when the sum of the differences over the dimensions is positive. In this example: $A - B = (10 - 5) + (9 - 7) + (8 - 6) = 5 + 2 + 2 = 9$, thus $A > B$; $B - C = (5 - 3) + (7 - 7) + (6 - 7) = 2 + 0 + (-1) = 1$, thus $B > C$; The example shows transitivity, since $A - C = (10 - 3) + (9 - 7) + (8 - 7) = 7 + 2 + 1 = 10 > 0$, thus $A > C$.

In general, the relation *greater than* applied to a phenomenon constituted of three events by n=3 dimensions shows transitivity.

	1	2	3
A	a1	a2	a3
B	b1	b2	b3
C	c1	c2	c3

$A - B = (a1 - b1) + (a2 - b2) + (a3 - b3) > 0$, then $A > B$; and $B - C = (b1 - c1) + (b2 - c2) + (b3 - c3) > 0$ then $A > C$.

Proof: Since $A > B$ $[(a1 + a2 + a3) > (b1 + b2 + b3)]$, and since $B > C$ $[(b1 + b2 + b3) > (c1 + c2 + c3)]$, where *ai*, *bi* and *ci* are real numbers, it follows that $A > C$ $[(a1 + a2 + a3) > (c1 + c2 + c3)]$ and transitivity is preserved. Clearly, the result is true for *A*, *B*, *C* and with n dimension, since the proof will be the same.

	1	2	3
A	1.0	0.1	0.1
B	0.5	1.0	0.3
C	0.1	0.3	0.5

Figure 2: Non-transitivity of multiplication ≥ 0.5

Transitivity will be preserved if we add weights (p, q, r) to the above three dimensions, where $0 < p < 1$, $0 < q < 1$, and $0 < r < 1$ and $p + q + r = 1$:

	1p	2q	3r
A	a1	a2	a3
B	b1	b2	b3
C	c1	c2	c3

$A > B$ if $p(a1 - b1) + q(a2 - b2) + r(a3 - b3) > 0$ and $B > C$ if $p(b1 - c1) + q(b2 - c2) + r(b3 - c3) > 0$; thus using the inner product we have $W \cdot A > W \cdot B$ and $W \cdot B > W \cdot C$, therefore $W \cdot A > W \cdot C$. Here $A = (a1, a2, a3)$, $B = (b1, b2, b3)$, $C = (c1, c2, c3)$ and $W = (p, q, r)$ and transitivity is preserved. Clearly, the result is true for A, B, C and W with n dimension since the proof will be the same.

In the next example, we will look at a case where transitivity is not preserved.

Figure 2 presents an example of the relation *multiplication ≥ 0.5* in which transitivity does not hold.

We define the relation *multiplication ≥ 0.5* when the sum of the multiplications over the dimensions ≥ 0.5. In table 2 we have $A \cdot B$=(1.0)(0.5) + (0.1)(1.0) + (0.10)(0.3) = 0.5 + 0.1 + 0.03 = 0.63 ≥ 0.5; $B \cdot C$ = (0.5)(0.1) + (1.0)(0.3) + (0.3)(0.5) = 0.05 + 0.3 + 0.15 = 0.5. However, $A \cdot C$ = (1.0)(0.1) + (0.1)(0.3) + (0.1)(0.5) = 0.1 + 0.03 + 0.05 = 0.18 < 0.5. That is, transitivity does not hold.

4 Acknowledgments

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Tiling Efflorescence of Expanding Kernels in a Fixed Periodic Array: Generalizing the Flower-Of-Life

Robert J. Marks II

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Abstract

Continually expanding periodically translated *kernels* on the two dimensional grid can yield interesting, beautiful and even familiar patterns. For example, expanding circular pillbox shaped kernels on a hexagonal grid, adding when there is overlap, yields patterns including maximally packed circles and a triquetra-type three petal structure used to represent the trinity in Christianity. Continued expansion yields the flower-of-life used extensively in art and architecture. Additional expansion yields an even more interesting emerging efflorescence of periodic functions. Example images are given for the case of circular pillbox and circular cone shaped kernels. Using Fourier analysis, fundamental properties of these patterns are analyzed. As a function of expansion, some effloresced functions asymptotically approach fixed points or limit cycles. Most interesting is the case where the efflorescence never repeats. Video links are provided for viewing efflorescence in real time.
Keywords: *tiling, emergence, periodicity, flower-of-life, efflorescence, triquetra*

1 Introduction

Expanding kernels on a periodic array can generate beautiful and sometimes familiar patterns.

To visualize how an expanding circular kernel might physically occur, imagine point sources of light that emit perfectly circular cones that expand with distance. This is illustrated in Figure 1. Assume a large number of these light sources are spaced in a hexagonal grid. A viewing screen placed parallel and close to the array of lights will display a set of circles as in Figure 2(a) because the cones have not yet overlapped. There will be a point where the cones first touch. This is illustrated in Figure 1 where a single hexagon of seven point elements sources are denoted by the small circles on top. The touching cones here correspond to the closely spaced nonoverlapping circles in Figure 2(b) and the maximally packed pennies in Figure 4. Beyond this point, the circles intersect. This visualization illustrates the dynamics in Figure 2 where the number of periodically spaced light sources on the plane is infinite. The further we go from the lights, the more circles overlap and the more interesting and beautiful patterns emerge.[1]

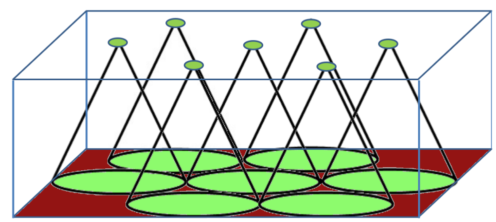


Figure 1: Illustration of the expanding kernels shown in Figure 2. Light sources give forth cone-shaped beams which overlap more and more as the distance is increased from the light source plane. Shown here is the point where the cones first touch. This corresponds to the circles in Figure 2(b).

From a first order approximation, expanding circles emerge from single x-rays in cone-beam tomography (Feldkamp, Davis, and Kress, 1984; Scarfe, 2018) generate the expanding circles in Figure 1. Expanding patterns other than cones naturally occur in electromagnetics. Periodically spaced antenna (Filipovic, Volakis, and Andersen, 1999; Ishimaru et al., 1985; Markov and Chaplin, 1983) and sensor arrays (Goussetis, Feresidis, and Vardaxoglou, 2006; Sung et al., 2008) generating identical expanding signals can display diverse and complex patterns depending on source excitation and range of observation.

The *tile* in this expanding circle example is a equalsided hexagons as used by bees in honey combs. The hexagonal tile for maximally packed circles is shown in Figure 5 where identical hexagonal tiles each containing an inscribed circle. Rectangles and parallelograms are other examples of possible

[1] As the cone expands, the light will grow dimmer. The analogy breaks down here. We assume the light in the plane always correspond to a brightness value of one no matter how far we are from the point sources.

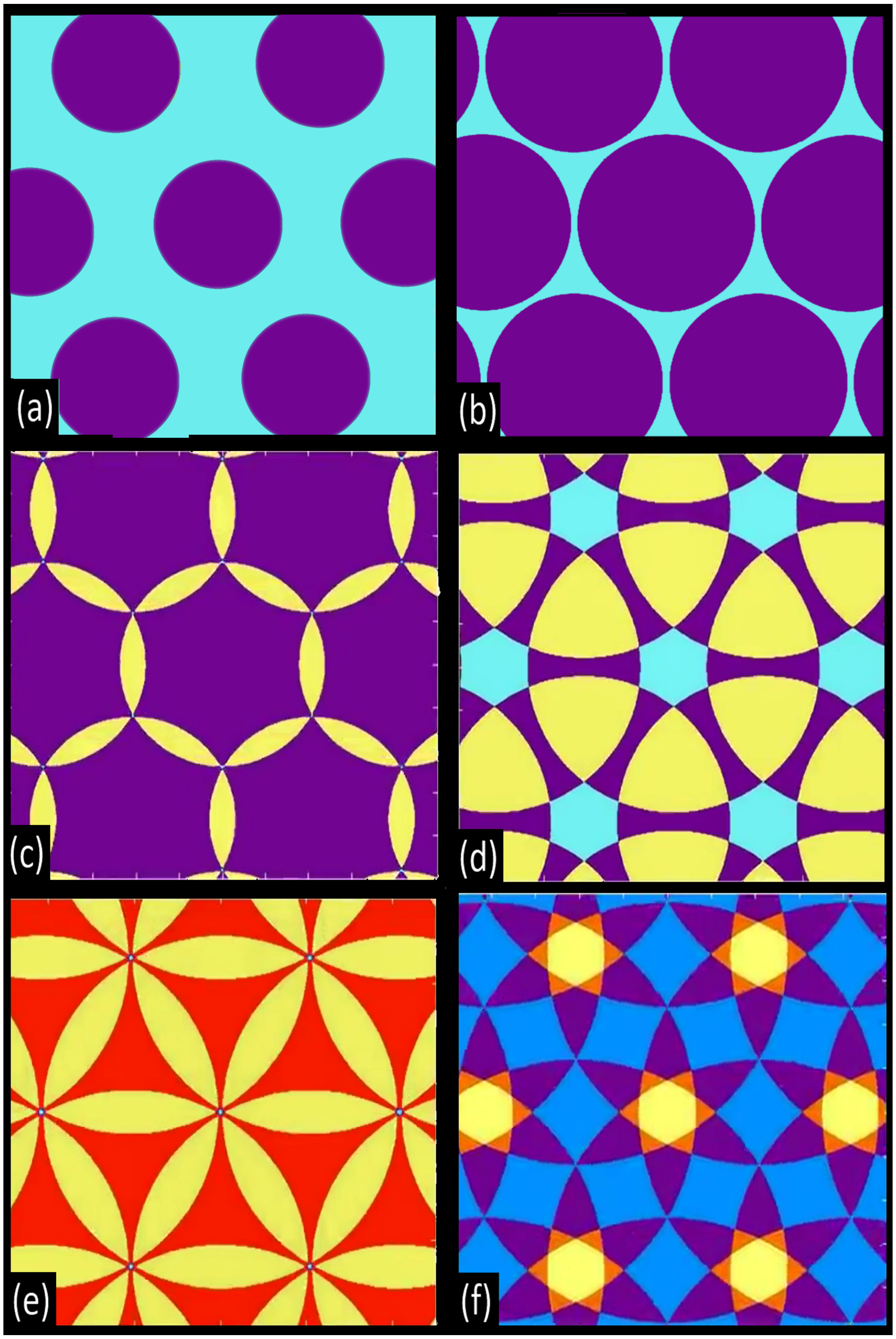
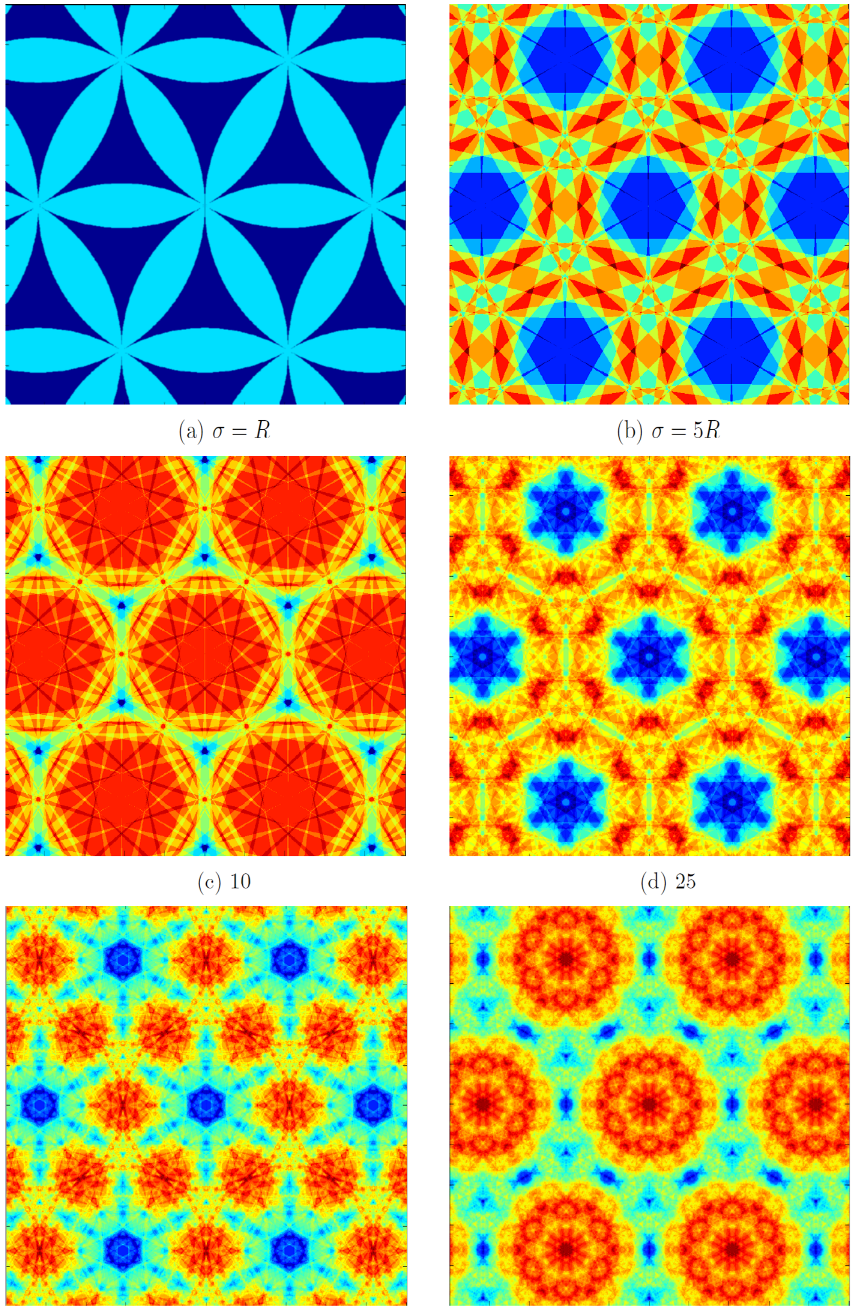


Figure 2: Expanding circular pillboxes on a hexagonal grid. Heat map (Wikipedia, 2020a) colours denote regions equal to a constant and are used only for differentiating among the different regions of the figure. Figure (b) represents maximally packed circles. Figure (c) is a three petal pattern. Slightly more expansion will result in the triquetra in Figure 7. Figure (e) is the flower-of-life. (Continued in Figure 3.)



(a) $\sigma = R$ (b) $\sigma = 5R$

(c) 10 (d) 25

(e) 50 (f) 100

Figure 3: Heat map plots of expanding circular pillbox efflorescent function beyond that in Figure 2. As the circles expand, the patterns have more texture because higher and higher spatial frequencies are introduced. (Continued from Figure 2.)

Figure 4: Maximally packed pennies (Wikipedia, 2020b). In this example, the kernel is the grey level map of a single penny and the periodicity is hexagonal.

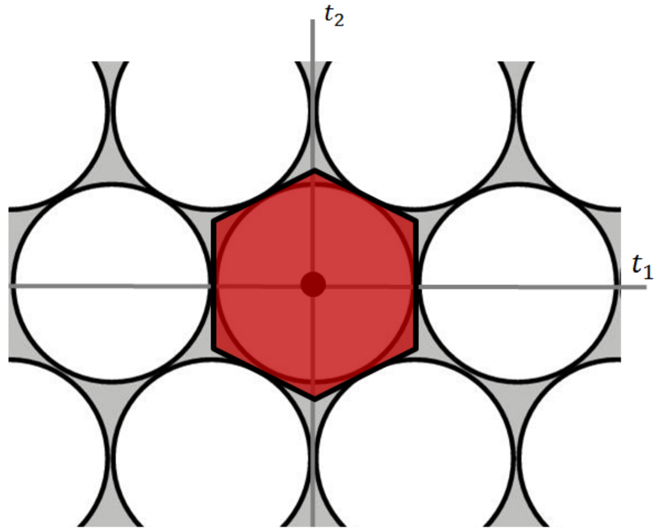


Figure 5: Maximally packed circles and the corresponding hexagonal tile.

tile shapes. Circles and octagons are not tiles since any tiling attempts will result in gaps of coverage.[2]

All the patterns in Figure 2 can be achieved with identically shaped hexagon tiles containing an appropriate pattern.

More generally, identical versions of a two dimensional (2D) function[3] are translated in accordance to a 2D periodic geometry specified by the two periodicity vectors. The translations are added to form a periodic function. We call the original 2D function the *kernel*. With the tiling geometry kept constant, each of these kernels is magnified or, equivalently, *expanded*. The expanding kernels will soon overlap onto other tiles. When kernels overlap, the expanded kernels are added. When viewing the emergent patterns as expansion continues, fascinating patterns can begin to flower. Since *efflorescence* in French means "to flower out," we refer to the emergence as *efflorescent functions*. No matter how much the kernels expand, the efflorescent function is a periodic function with a period fixed by the tile.

An alternate explanation of expanding kernels is illustrated in Figure 1 where the circular kernel is illustrated as an expanding circular pill box.

Independent of the degree of kernel expansion, the resulting

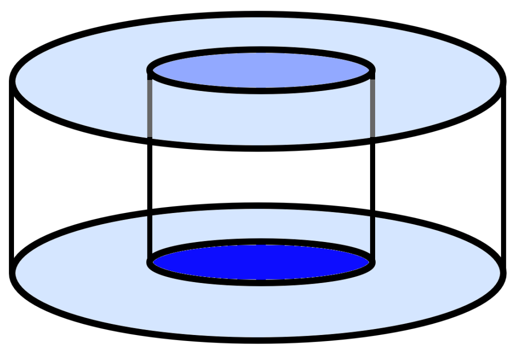


Figure 6: An expanding circular pillbox as a 2D function. The circular pillbox shape expands to the larger pillbox is a smooth continuous manner.

[2]Triangles are not considered tiles in our treatment. Two identical triangles can be be configured into a parallelogram tile, but we we only consider tiling that uses translation. No rotation or flipping is allowed. Nevertheless, rotation and flipping of so-called *subtiles* can be used in definition multidimensional periodicity (Marks II, 2019). This is not considered in our treatment.

[3]i.e. a scalar function of two variables.

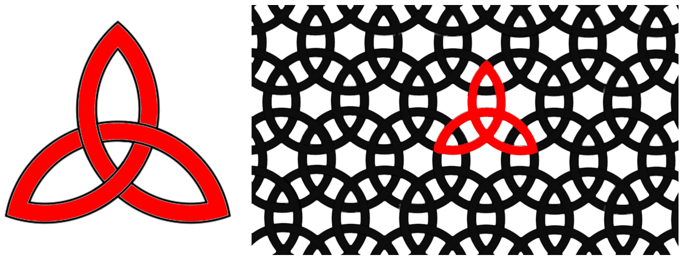


Figure 7: An illustration of a specific instance of expanded overlapping when the kernel is an annulus (circular ring) replicated hexagonally. Left: The triquetra (Wikipedia, 2020c). Right: Periodic replication of the triquetra reveals emergence from overlapping rings. (Note: The summation of the overlaps is not shown. Only the ring overlap is shown.)

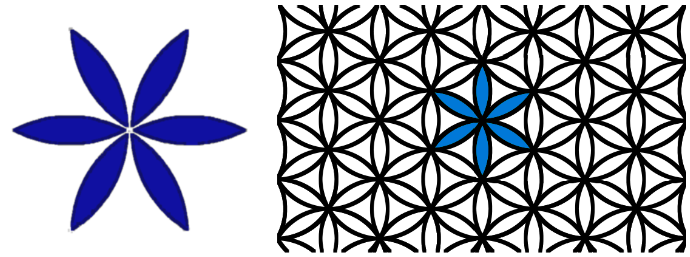


Figure 8: Left: The flower-of-life. Right: Periodic replication of the flower-of-life reveals emergence from overlapping circles.

2D function formed by the sum of possibly overlapping kernels is periodic with the same hexagonal replication geometry. The single hexagonal tile shown in Figure 5can be copied and, if you will, used to tile a kitchen floor without gaps. In this sense, a periodic tiling geometry is conserved during the expansion.[4]

Expanding circles generate familiar flowering instances including:

- packed circles in Figure 2(b) can be visualized by placing pennies on a table surface as close as possible. This is illustrated in Figure 4,
- as seen in Figure 7, a triquetra-type (Wikipedia, 2020c) three petal pattern used to represent the trinity in early Christianity. This pattern is seen in Figure 2(c), and
- the flower-of-life (Melchizedek, 1999) in Figure 2(e)[5]

Properties of efflorescent functions can be derived using two dimensional Fourier series analysis. Some efflorescent functions approach fixed points or limit cycles as a function of expansion. Most interesting are efflorescent functions that never repeat. Describing mathematics is limited to Section 4 and the Appendix. The content of the paper is thereby accessible without reference to the detail sought by mathematicians for deeper insight.

[4]For a given replication, there is more than one choice for a tile. Hexagonal tiles can be used for any of the patterns shown in Figure 2 but a parallelogram tile can be used to represent the same periodicity structure. Although tiles can differ geometrically, all viable tiles will have the same two-dimensional area (Marks II, 2009).

[5]The flower-of-life is easily constructed using only a compass. Draw a circle. Then place the point of the compass at any point on the circle. Draw another circle. Place the compass point at one of the two points where the circles intersect and draw another circle with the same radius. Continue placing the compass point at intersections of circles with the original circle and the flower-of-life will result.

1.1 The Flower-of-Life in Art History

The flower-of-life tiling has a rich history in art and architecture. The flower-of-life appears in the Osiris Temple in ancient Egypt (Flowers, 2006) and as a floor decoration from the palace of King Ashurbanipal. Ashurbanipal was king of the Neo-Assyrian Empire from 668 BC to c. 627 BC (Manninen, 2011). The flower-of-life even appears in crop circles (National Geographic, 2010) .

Here are some other examples.

1. As shown in Figure 10, the flower-of-life appeared in the art of Leonardo da Vinci (Mic, 2012).
2. Figure 10 shows the flower-of-life from Turkey.
3. In Figure 11, we see a cup fragment from Idalion, Cyprus that dates to circa 700 to 600 BC. The art shows "mythological scenes, a sphinx frieze and the representation of a king vanquishing his enemies. The center contains a version of the 'Flower of life' geometrical pattern" (Nguyen, 2007).
4. A ball "held by the male Imperial Guardian Lion at the Gate of Supreme Harmony, Forbidden City, Beijing. China" is covered by replications of the flower-of-life. This is shown in Figure 12.

1.2 Further Circle Expansion

What happens when the circles are expanded further than shown in Figure 2? Results is shown in Figure 3 starting with the flower-of-life in the upper left. Assume the original radius of the circles in the flower-of-life is $R = 1$. Shown are hexagonal tilings corresponding to the circle radiuses of 5, 10, 25, 50 and 100.

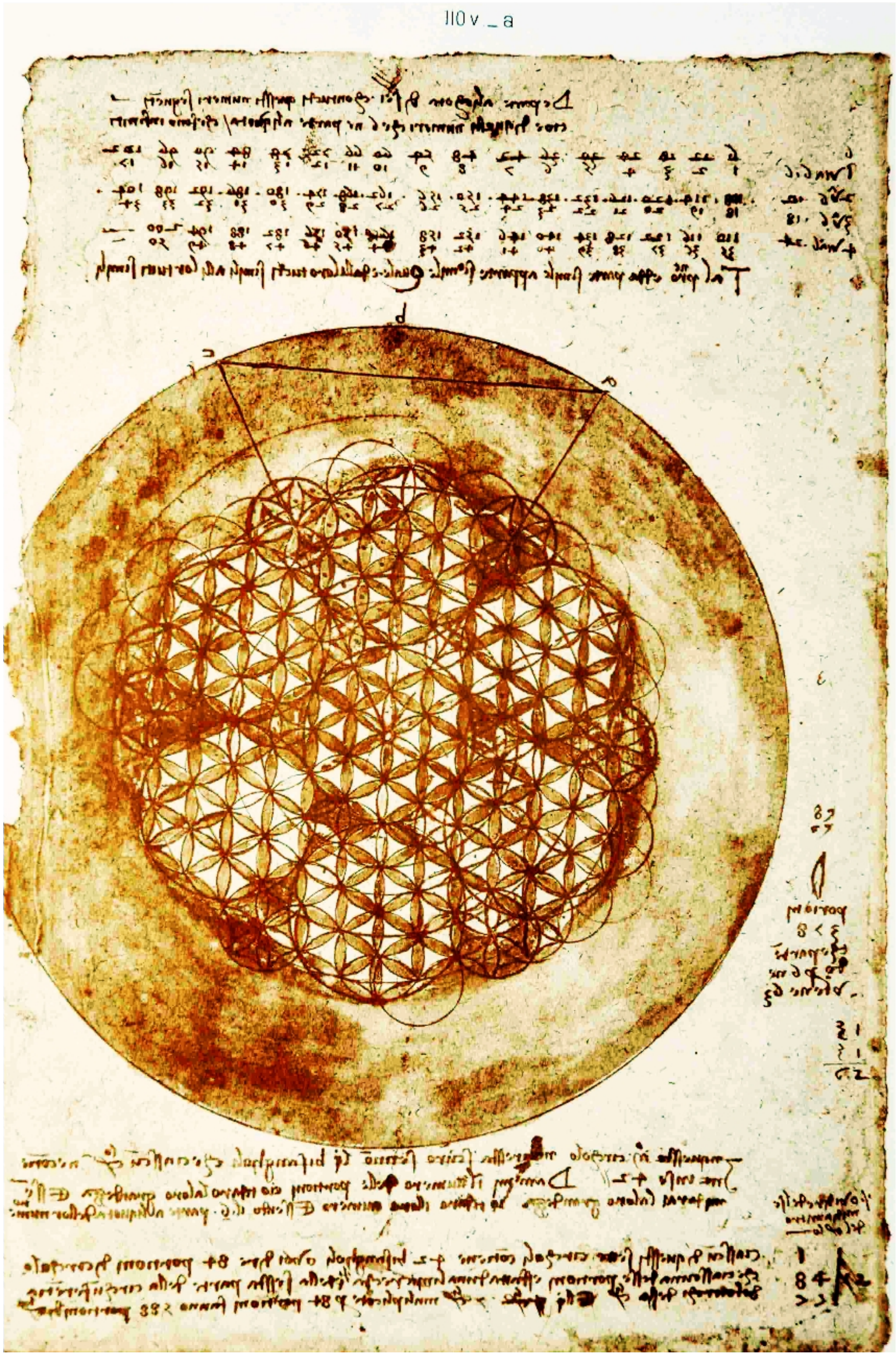


Figure 9: Flower of life from sketches from Leonardo da Vinci. (This is a faithful photographic reproduction of a two-dimensional, public domain work of art (da Vinci, 1478).)

Figure 10: Flower of life "ephesos square" from Ephesus, Turkey. (Image credit: Wikipedia Creative Commons Attribution-Share Alike 4.0 International (Miryam, 2015).)

Figure 11: An ancient cup inscibed with the flower-of-life. (Image credit: This work is in the public domain in its country of origin and other countries and areas where the copyright term is the author's life plus 100 years or fewer (Nguyen, 2007).)

Figure 12: The flower-of-life is on a ball at the Gate of Supreme Harmony, Forbidden City, Beijing. China. (Photo credit: Wikimedia Commons (Adamantios, 2013).)

Videos of beautiful emergent behavior from real-time expanding circular kernels are available online (Nybal, 2014d; Nybal, 2014c).

2 Other Expanding Kernels

Other kernels can be expanded and arrays other than hexagonal can be used. An expanding circular cone kernel[6] is illustrated in Figure 13. (Compare to the expanding pillbox circle in Figure 6.) A heat map plot of a hexagonal array of small nonoverlapping circular cones is shown in Figure 14(a). The overlapping expanding kernels are then shown for circle radii of $200, 500, 1000, 1500$, and 1750. As is the case with the circles, the patterns generally become more complex as the expansion becomes larger and more and more cones intersect.

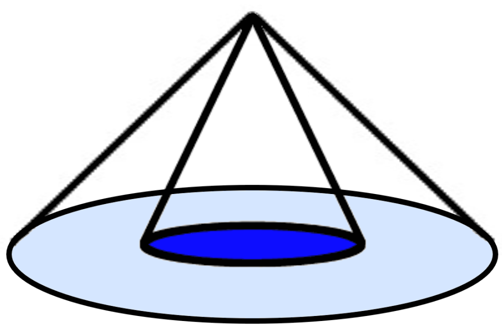


Figure 13: A single expanding circular cone shaped kernel as a two dimensional function.

3 Properties of Expanding Kernels of Varying Periodicity

3.1 Detrending and Heat Maps

As more and more kernels overlap, the number of kernels intersecting a tile generally gets larger and larger.[7] If 1000 circular pillboxes overlap and the height of a pillbox is one, the value of the 2D function is 1000 in the area of overlap. Detrending clears this tower by removing the buildup and looking only at fluctuations on top of the tower after removing the tower height.

In Fourier series, the zeroth order Fourier series coefficient denotes the average value of the periodic function. By setting the zeroth order coefficient to zero, the tower is removed and only the fluctuations remain. We define setting the zeroth order Fourier series coefficient to zero as *detrending* (Hill and Gauch, 1980; Kantelhardt et al., 2002). The heat map plots in Figures 2,3,14 and 15 do this automatically by plotting only within the dynamic range of the fluctuations.

3.2 Summary of Fourier Analysis Results

Depending on the kernel and periodicity, display of detrended patterns show different behaviour. In Section 4, we examine whether continuous expansion of kernels asymptotically approaches either:

1. zero everywhere,
2. a fixed periodic function of the $\vec{t}$ plane that does not change with respect to additional expansion, or
3. oscillation in a limit cycle as function of the expansion variable. In other words, as expansion continues, the efflorescent function displays a repeated pattern.

As we go down the list, each entry is seen to be a subset of the other. A value of (1) zero is a degenerate case of (2) a fixed periodic function that does not change with expansion. Likewise (2), a fixed periodic function, is a special static case of (3): oscillation on a limit cycle as a function of expansion.

The most interesting cases, not on the list, are those where the efflorescent function never repeats and results in a never repeating series of patterns. The expanding circular pillbox and circular cone are examples.

4 Analysis

The mathematical analysis of efflorescent functions is solely relegated to this section and the Appendix.

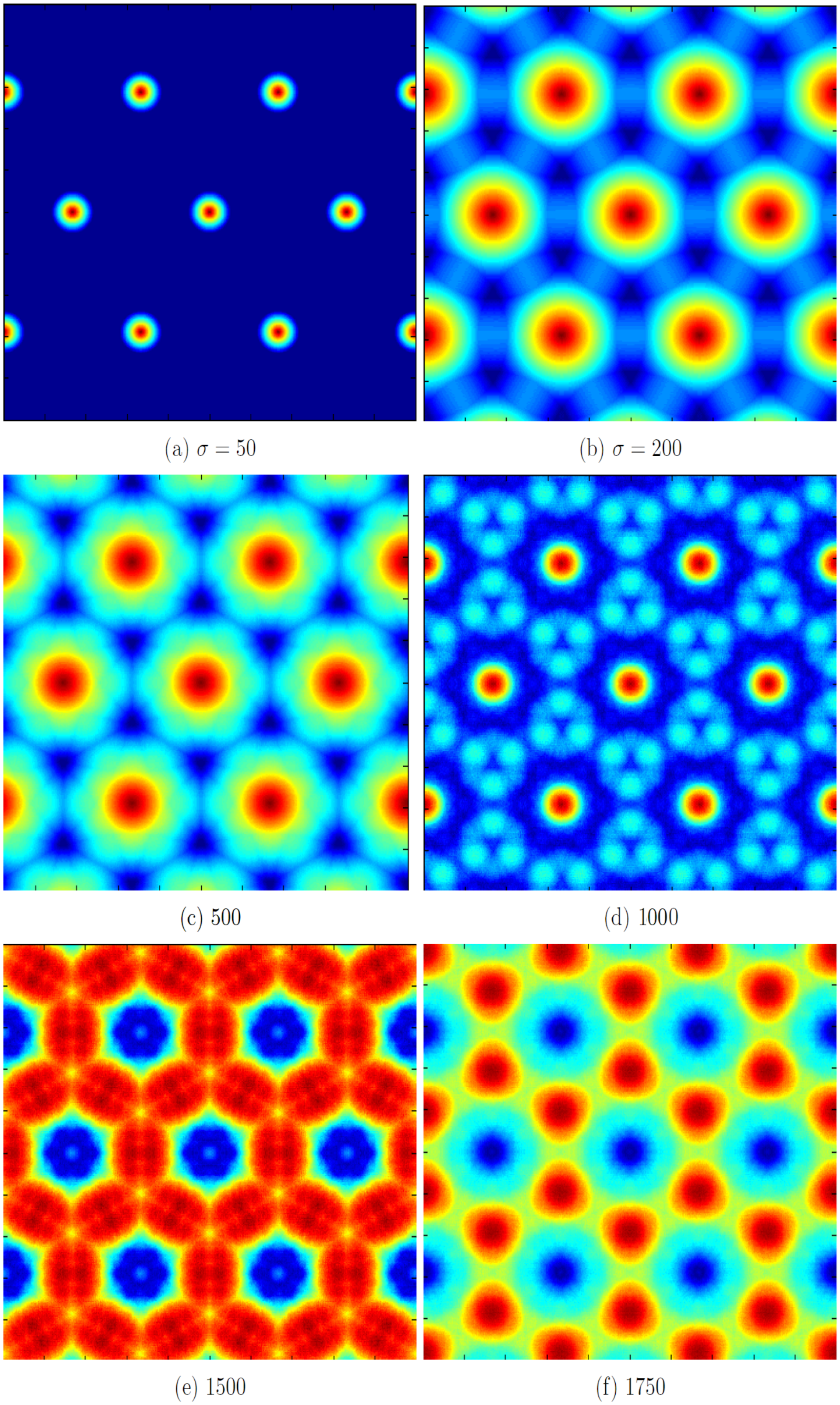
Using standard notation (Dudgeon and Mersereau, 1984; Marks II, 1991; Marks II, 2009), the two dimensional Fourier transform of a two dimensional function $x(\vec{t}) = x(t_1, t_2)$ is[8]

$$X(\vec{u}) = \int_{\vec{t}} x(\vec{t}) e^{-i2\pi\vec{u}^T\vec{t}} d\vec{t} \quad (1)$$

[6]Disambiguation: The term *cone-kernel* is also used in reference to 2D time-frequency representations (Oh and Marks II, 1992; Oh, Marks II, and Atlas, 1994; Zhao, Atlas, and Marks II, 1990) and is not related to the usage of the term here.

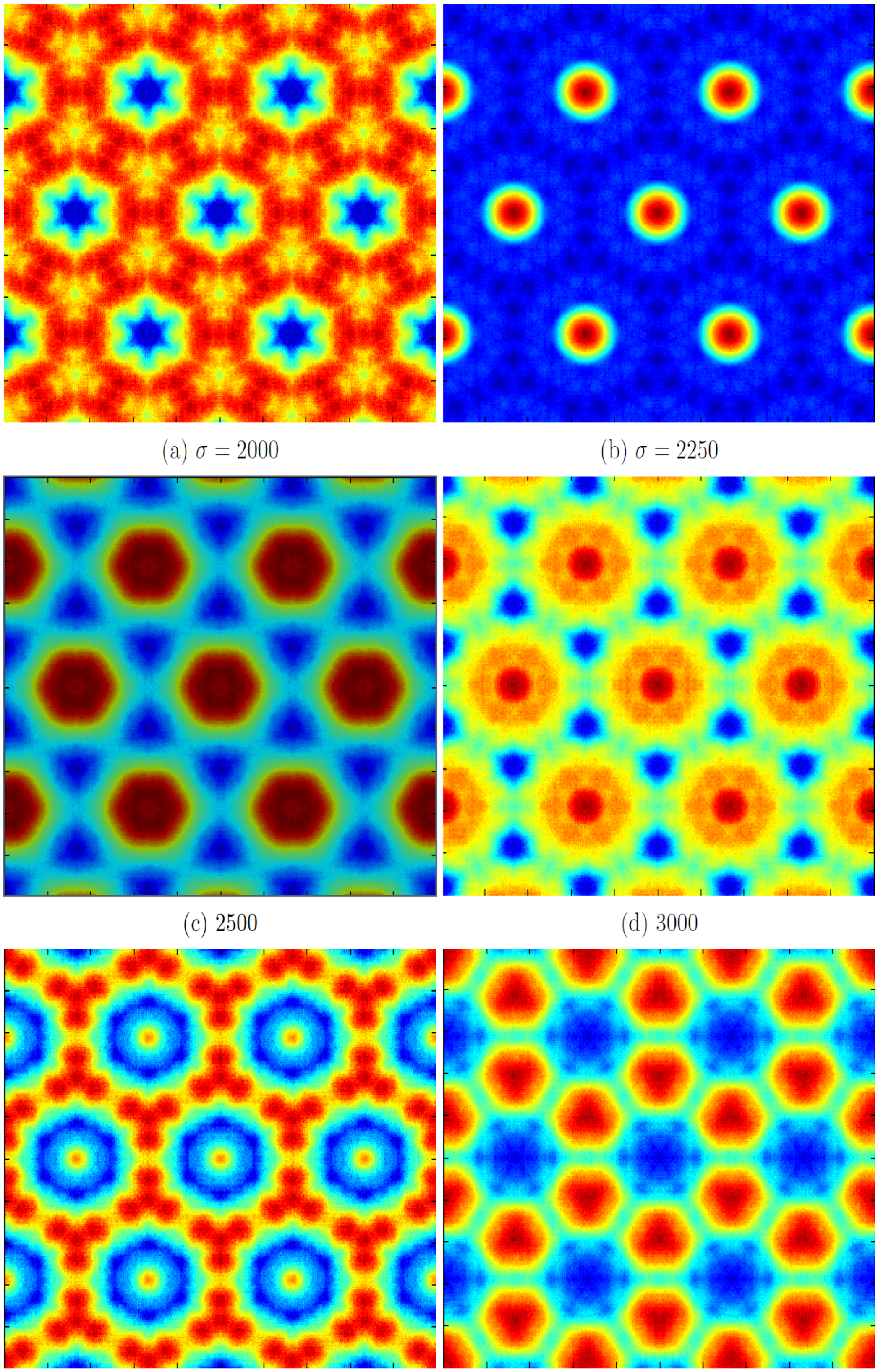
[7]An exception would be a gaussian shaped kernel where all tiles are effected by all other tiles at all times. The contribution of shifted kernels far removed will become more and more significant as the kernel expands.

[8]$i = \sqrt{-1}$.



(a) $\sigma = 50$ (b) $\sigma = 200$ (c) 500 (d) 1000 (e) 1500 (f) 1750

Figure 14: Heat map plots of the expanding circular cone efflorescent function. The nonoverlapping small cones are seen in (a). (Continued in Figure 15.)



(a) $\sigma = 2000$ (b) $\sigma = 2250$

(c) 2500 (d) 3000

(e) 3250 (f) 3750

Figure 15: Heat map plots of the expanding circular cone efflorescent function. (Continued from Figure 14.)

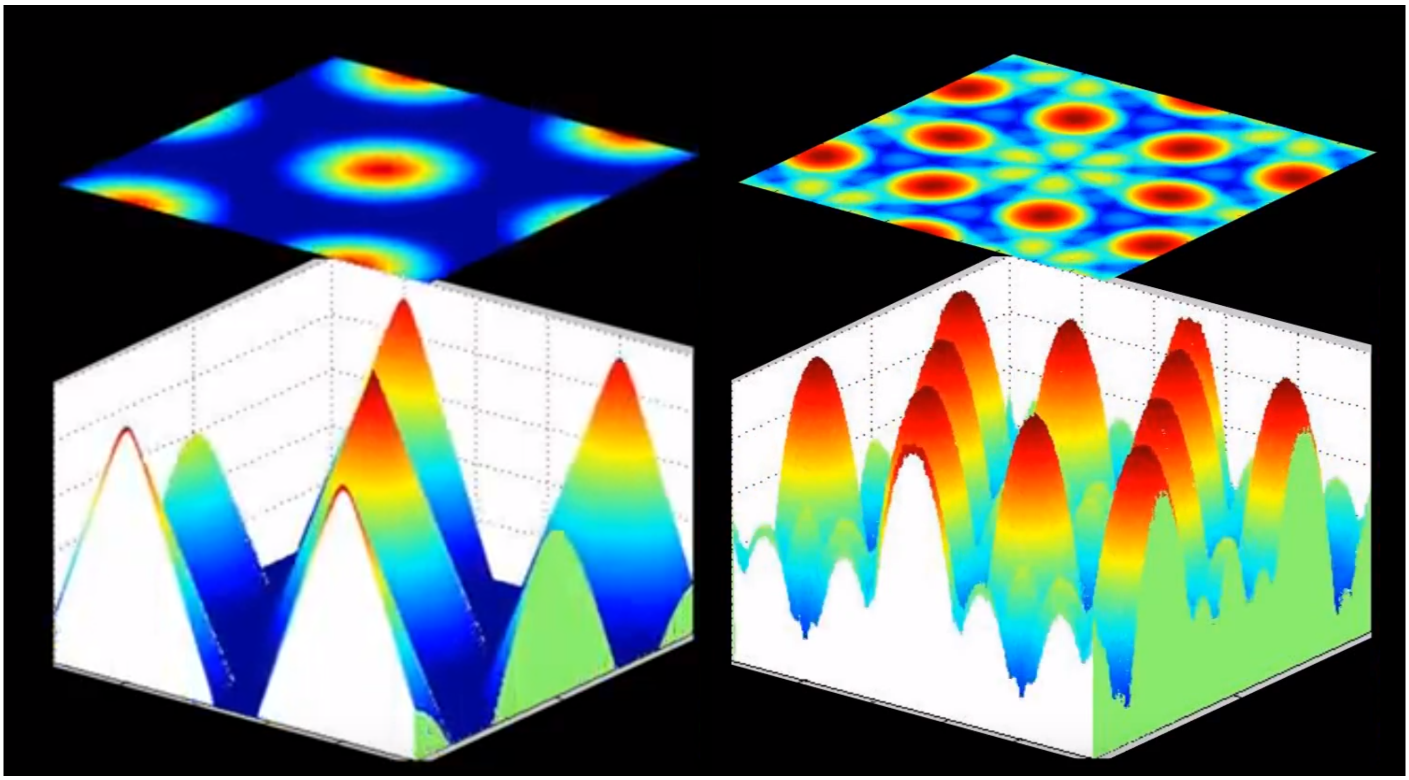


Figure 16: Left: A 2D plot of nonoverlapping replicated circular cones. A perspective projection of the heat map is shown on top. Right: A similar plot made after the expanding circular cones intersect.

where T denotes vector transposition, $\vec{t} = [t_1, t_2]^T$, $\vec{u} = [u_1, u_2]^T$, $d\vec{t} = dt_1 dt_2$ and

$$\int_{\vec{t}} = \int_{t_1}\int_{t_2}.$$

The *signal integral property* (Marks II, 2009) follows immediately from (1) by setting $\vec{u} = \vec{0}$.

$$X(\vec{0}) = \int_{\vec{t}} x(\vec{t})d\vec{t}. \tag{2}$$

The inverse Fourier transform is

$$x(\vec{t}) = \int_{\vec{u}} X(\vec{u})e^{i2\pi\vec{u}^T\vec{t}}d\vec{u}.$$

To supply foundation and to establish notation, a concise review of multidimensional Fourier series is appropriate (Dudgeon and Mersereau, 1984; Marks II, 1991; Marks II, 2009). The Fourier series has the following properties (Marks II, 2009):

- Convergence is in the mean if a period of the periodic function satisfies *Dirichlet conditions* criteria.
- Convergence is uniform if the periodically replicated kernel is continuous.
- When there are discontinuities in the kernel, the Fourier series converges to the arithmetic midpoint of the discontinuity.

The examples in this paper are for one and two dimensional periodic functions although the theory can be developed for an arbitrary dimension.

Two dimensional periodicity is dictated by a 2×2 nonsingular periodicity matrix $\mathbf{Q}$ given as

$$\mathbf{Q} = [\vec{q}_1\vec{q}_2]$$

where $\vec{q}_1$ and $\vec{q}_2$ are *periodicity vectors*. In one dimension, the period of a periodic function is defined by a single scalar which can be viewed as a 1×1 matrix. The scalar entry in the matrix is the one dimensional period, T. In two dimensions, a pair of 2D vectors is required to define periodicity. In M dimensions, M periodicity vectors are required. Each vector is of length M (Dudgeon and Mersereau, 1984; Marks II, 1991; Marks II, 2009),

A 2D example is shown in Figure 17 where maximally packed circles of radius R generate periodicity vectors

$$\vec{q}_1 = \begin{bmatrix} \frac{R}{2} \\ \frac{R}{2\sqrt{3}} \end{bmatrix};\ \vec{q}_2 = \begin{bmatrix} -\frac{R}{2} \\ \frac{R}{2\sqrt{3}} \end{bmatrix}.$$

where R is the circle's radius. The corresponding *periodicity matrix* follows as

$$\mathbf{Q} = \frac{R}{2}\begin{bmatrix} 1 & -1 \\ \sqrt{3} & \sqrt{3} \end{bmatrix}. \tag{3}$$

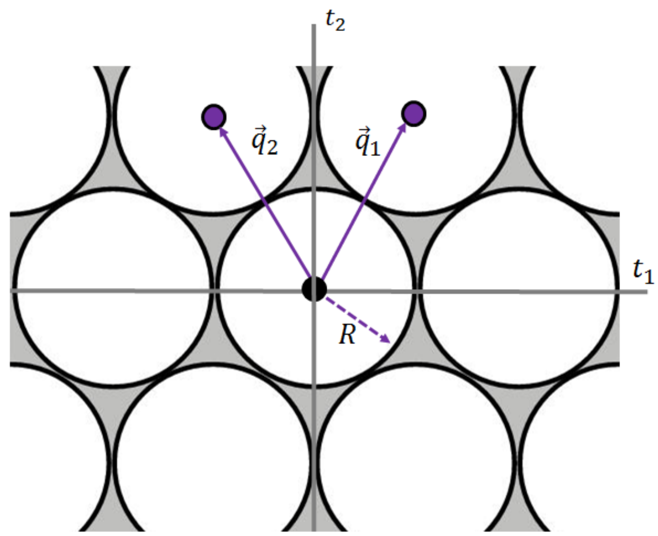


Figure 17: Hexagonal periodicity vectors illustrated for maximally packed circles.

A tile isolates a single period of the periodic function and, when replicated according to the periodicity matrix, fills the space without gaps. For hexagonal periodicity, a corresponding hexagonal tile is shown in Figure 5. For a given periodicity structure, neither $\mathbf{Q}$ or the tile shape is unique. This is illustrated Figure 18 where hexagon and a parallelogram tiles both have the same periodicity vectors.

A tile centered at the origin will be replicated on the (t_1, t_2) plane at the vectors $\vec{q}_1$ and $\vec{q}_2$. The tile will also be replicated at any integer multiple of the periodicity vectors, for example at $\vec{q}_1 + \vec{q}_2$ and $4\vec{q}_1 - 3\vec{q}_2$. Any tile replication on the (t_1, t_2) plane can be represented by the combination $m_1\vec{q}_1 + m_2\vec{q}_2$ where m_1 and m_2 are integers. A more concise expression is

$$m_1\vec{q}_1 + m_2\vec{q}_2 = \mathbf{Q}\vec{m}$$

where $\vec{m}$ is a two dimensional vector of integers.

$$\vec{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}.$$

The hexagonally shaped tile in Figure 5 has an area of

$$|\det \mathbf{Q}| = \frac{\sqrt{3}\,R^2}{2}. \tag{4}$$

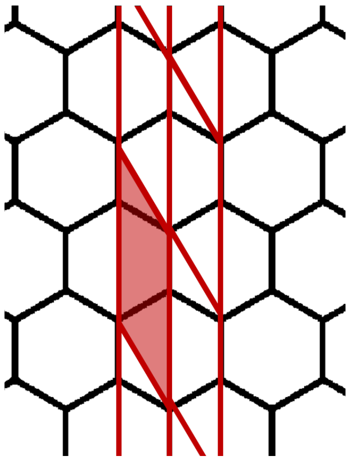


Figure 18: For a given set of periodicity vectors, the choice of tiles is not unique. The periodicity vectors illustrated at the bottom of Figure 17 can also describe the parallelogram tile shown here. (Only two columns of the parallelogram tile are shown here.) In both cases, the area of the tile, $|\det Q|$, is the same (Marks II, 2009).

For a given periodicity structure defined by the periodicity matrix $\mathbf{Q}$, a periodic function with a kernel of $g(\vec{t})$ can be written in a space with coordinates $\vec{t}$ as (Marks II, 2009)

$$z(\vec{t}) = \sum_{\vec{m}} g(\vec{t} - \mathbf{Q}\vec{m}) \tag{5}$$

where the sum is over the set of all integer pairs.

$$\sum_{\vec{m}} = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} .$$

Note that

- the kernel is not constrained to be zero outside of a tile and can even extend over the entire $\vec{t}$ plane, and
- many kernels can generate the same periodic function, $z(\vec{t})$.

The corresponding multidimensional Fourier series of the periodic function in (5) is the Fourier series (Marks II, 2009; Papoulis, 1978)[9]

$$z(\vec{t}) = |\det \mathbf{P}| \sum_{\vec{m}} G(\mathbf{P}\vec{m}) \exp\left(i2\pi\vec{t}^T \mathbf{P}\vec{m}\right) \tag{6}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are related by an inverse transpose

$$\mathbf{P} = \mathbf{Q}^{-T}, \tag{7}$$

The equivalence of (5) and (6) stems from the Fourier dual of the *Poisson sum formula* (Marks II, 2009; Papoulis, 1978; Papoulis and Pillai, 2002).

$$\sum_{\vec{n}} X(\vec{u} - \mathbf{P}\vec{n}) = |\det \mathbf{Q}| \sum_{\vec{n}} x(\mathbf{Q}\vec{n})\, e^{i2\pi\vec{u}^T \mathbf{Q}\vec{n}}$$

4.1 Expanding Kernels

We are able to now describe the expanding kernel periodic function for arbitrary periodicity matrix $\mathbf{Q}$ and kernel $g(\vec{t})$.

Definition 4.1. The *expanding kernel periodic function*, $x_\sigma(\vec{t})$, generated by a *kernel* $g(\vec{t})$ is

$$x_\sigma(\vec{t}) = \sum_{\vec{m}} g\left(\frac{\vec{t} - \mathbf{Q}\vec{m}}{\sigma}\right) \tag{8}$$

As σ increases, the kernel expand.

In both the expanding circular pillbox and expanding circular cone examples, σR is the radius of the circle.

From (6), the corresponding Fourier series of the expanding kernel is

$$x_\sigma(\vec{t}) = |\det \mathbf{P}|\sigma^2 \sum_{\vec{m}} G(\sigma\mathbf{P}\vec{m}) \exp\left(i2\pi\vec{t}^T \mathbf{P}\vec{m}\right). \tag{9}$$

The overlapping expanding kernels can be detrended by evaluating the mean value of the periodic function. The mean value in a Fourier series expansion is the zeroth order Fourier series coefficient. This can be evaluating by integrating over a single tile followed by division by the area of the tile.

The theorem to follow uses the arbitrariness of the choice of tiles when integrating. In one dimension, the period of a periodic function, say T, is arbitrary. We can choose the period to be on the interval $0 \le t < T$ or $-T/2 \le t < T/2$. There is a

[9] To see the periodicity, consider shift of (6) from $z(\vec{t})$ to $z\left(\vec{t} - \mathbf{Q}\vec{k}\right)$ where $\vec{k}$ is an arbitrary vector of integers. If $z\left(\vec{t} - \mathbf{Q}\vec{k}\right) = z(\vec{t})$ for all such shifts. $z(\vec{t})$ is periodic with periodicity matrix $\mathbf{Q}$. From (6),

$$z\left(\vec{t} - \mathbf{Q}\vec{k}\right) = |\det \mathbf{P}| \sum_{\vec{m}} G(\mathbf{P}\vec{m}) \exp\left(i2\pi\left(\vec{t} - \mathbf{Q}\vec{k}\right)^T \mathbf{P}\vec{m}\right).$$

The exponential term here becomes

$$\exp\left(j2\pi\left(\vec{t} - \mathbf{Q}\vec{k}\right)^T \mathbf{P}\vec{m}\right) = \exp\left(i2\pi\vec{t}^T \mathbf{P}\vec{m}\right) \exp\left(-i2\pi\left(\mathbf{Q}\vec{k}\right)^T \mathbf{P}\vec{m}\right).$$

Using (7),

$$\exp\left(-i2\pi\left(\vec{k}^T \mathbf{Q}^T \mathbf{P}\vec{m}\right)\right) = \exp\left(-i2\pi\left(\vec{k}^T \vec{m}\right)\right) = 1$$

so that $z\left(\vec{t} - \mathbf{Q}\vec{k}\right) = z(\vec{t})$.

similar arbitrariness in the choice of a tile. To illustrate, consider a 2D function with hexagonal replication across a plane. Choosing a tile in this function is like choosing a cookie cutter. As illustrated in Figure 18, a hexagonal cookie cutter can be used. But a hexagon is not the only possible tile. As shown in the figure, a cookie cutter shaped like a parallelogram can also be used as can any cookie cutter that satisfies the periodicity constraints.

Notationally, integrating over area C in the following theorem means integration over any single tile. For additional details, see Marks (Marks II, 2009).

Theorem 4.2. *The mean value of the expanding kernel function, $x_\sigma(\vec{t})$, is*

$$\begin{aligned} \langle x_\sigma \rangle &:= \frac{1}{|\det \mathbf{Q}|} \int_{\vec{t}\in C} x_\sigma(\vec{t})\, d\vec{t} \\ &= \sigma^2 |\det \mathbf{P}| \int_{\vec{t}} g(\vec{t})\, d\vec{t} \qquad (10) \\ &= \sigma^2 |\det \mathbf{P}| G(\vec{0}) \qquad (11) \end{aligned}$$

The region C is any region in the $\vec{t}$ space covering a tile.

As σ increases, the detrended sum of the expanding kernels in the $\vec{t}$ plane approaches a without any interesting structure.

Proof. The expression in (10) for the mean of $x_\sigma(\vec{t})$ follows from the $\vec{m} = \vec{0}$ term in the Fourier series in (9). Equation (11) follows from the integral property (2). □

We can now define the detrended periodic function.

Definition 4.3. The periodic *efflorescent function*, $\zeta_\sigma(\vec{t})$, is the detrended expanding kernel function, *i.e.* $x_\sigma(\vec{t})$ minus its mean.

$$\zeta_\sigma(\vec{t}) = x_\sigma(\vec{t}) - \langle x_\sigma \rangle .$$

The corresponding Fourier series of $\zeta_\sigma(\vec{t})$ is simply the Fourier series of $x_\sigma(\vec{t})$ in (9) with the zeroth order Fourier series coefficient $\vec{m} = \vec{0}$ term removed.[10]

$$\zeta_\sigma(\vec{t}) = |\det \mathbf{P}| \sigma^2 \sum_{\vec{m}\neq\vec{0}} G(\sigma \mathbf{P}\vec{m}) \exp\left(i2\pi \vec{t}^T \mathbf{P}\vec{m}\right). \qquad (12)$$

From this expression, we see the Fourier series coefficients for the efflorescent function are

$$c_\sigma[\vec{m}] = \begin{cases} |\det \mathbf{P}| \sigma^2 G(\sigma \mathbf{P}\vec{m}) & ; \quad \vec{m} \neq \vec{0} \\ 0 & ; \quad \vec{m} \neq \vec{0} \end{cases} \qquad (13)$$

[10]By $\vec{0}$, we mean a vector whose only elements are zero.

4.2 Convergence

The periodic efflorescent function can most interestingly generate a never repeating pattern of fascinating shapes. This does not happen when, as a function of expansion, the efflorescent function reaches a limit cycle or fixed point. We now examine when this happens.

Asymptotic Convergence of the Efflorescent Function to Zero

We first establish when the efflorescent function expanding kernel approaches zero.

First define the *Kronecker delta* as

$$\delta[\vec{m}] := \begin{cases} 1 & ; \quad \vec{m} = \vec{0} \\ 0 & ; \quad \vec{m} \neq \vec{0} \end{cases}$$

Theorem 4.4. Sufficient condition for converging to the mean. *Let $M = 2$.*[11] *If*

$$\sigma^M G(\sigma \mathbf{P}\vec{m}) \underset{\sigma\to\infty}{\longrightarrow} \sigma^M G(\vec{0}) \delta[\vec{m}] \qquad (14)$$

then $x_\sigma(\vec{t})$ converges to its mean.

$$x_\sigma(\vec{t}) \underset{\sigma\to\infty}{\longrightarrow} \langle x_\sigma \rangle .$$

As a consequence

$$\zeta_\sigma(\vec{t}) \underset{\sigma\to\infty}{\longrightarrow} 0.$$

In other words, as the kernels continue to expand, the efflorescent function approaches the very uninteresting result of zero over the entire (t_1, t_2) plane.

Proof. As $\sigma \to \infty$, all terms $\sigma^M G(\sigma \mathbf{P}\vec{m})$ in (9) tend to zero when (14) is true except when $\vec{m} = \vec{0}$. Then (9) becomes

$$x_\sigma(\vec{t}) \underset{\sigma\to\infty}{\longrightarrow} \sigma^M |\det \mathbf{P}| G(\vec{0}) = \langle x_\sigma \rangle$$

□

A sufficient smoothness criterion Convergence of an expanding efflorescent function to zero is assured when the kernel adheres to smoothness and integrability properties.

[11]The theorems given are applicable in any dimension M. We have concentrated on 2D so will set $M = 2$ to avoid confusion. This also applies to Theorem 4.5.

Theorem 4.5. Convergence to the mean: *The following theorem applies to M dimensions. For the examples herein, $M = 2$. The M dimensional function, $x_\sigma(\vec{t})$, converges to its mean if its kernel, $g(\vec{t})$, obeys the following property.*[12]

$$\int_{\vec{t}} \left| \left(\prod_{k=1}^{M} \frac{\partial^{i_k}}{\partial t_k^{i_k}} \right) g(\vec{t}) \right| d\vec{t} = A < \infty \tag{15}$$

where the nonnegative integers $\{i_k | 1 \le k \le M\}$ obey

$$\sum_{k=1}^{M} i_k > M \tag{16}$$

The choice of $\vec{i} = [i_1\ i_2\ i_3\ \cdots\ i_M]^T$ is arbitrary so long as (15) *and* (16) *are satisfied.*

The proof is given in Appendix 6.1.

Corollary 4.6. *For $M = 1$, Theorem 4.5 says the efflorescent function will converge to zero if*

$$\int_t \left| \frac{d^2}{dt^2} g(t) \right| dt = A < \infty$$

Example 4.7. Consider the two dimensional kernel

$$g(\vec{t}) = \Pi(t_1) e^{-t_2^2},$$

where the *rectangle function* is

$$\Pi(t) := \begin{cases} 1 & ; \quad |t| \le \frac{1}{2} \\ 0 & ; \quad |t| > \frac{1}{2}. \end{cases} \tag{17}$$

Then (WolframAlpha.com, 2020)

$$\begin{aligned} \int_{\vec{t}} \left| \frac{\partial^3}{\partial t_2^3} g(\vec{t}) \right| d\vec{t} &= \int_{t_1=-\infty}^{\infty} \Pi(t_1) dt_1 \int_{t_2=-\infty}^{\infty} \left| \frac{d^3}{dt_2^3} e^{-t_2^2} \right| dt_2 \\ &= \\ &= 4\left(1 + 4e^{-3/2}\right) = A < \infty \end{aligned}$$

and $i_1 + i_2 = 0 + 3 > M = 2$. The criteria in Theorem 4.5 are met and asymptotic convergence of the efflorescent function to zero is assured.

Asymptotic Convergence To a Fixed Function

As the following example shows, the efflorescent function can converge to a fixed function of $\vec{t}$ as scaling increases.

Example 4.8. Consider the one dimensional kernel

$$g(t) = e^{-t} \mu(t) \tag{18}$$

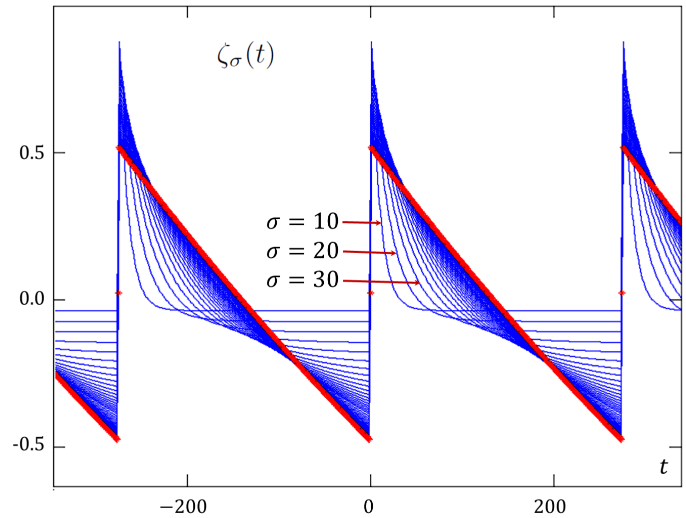


Figure 19: Using the exponential decay kernel in (18) results in the $\zeta_\sigma(t)$'s shown. As $\sigma \to \infty$, the function approaches the sawtooth shown by the thick red line given by (19). (For this plot, $T = 275$.)

where $\mu(t)$ is the Heaviside step function.[13] Then $\zeta_\sigma(\vec{t})$ converges in steady state to the solid red sawtooth waveform shown in Figure 19.

$$\lim_{\sigma \to \infty} \zeta_\sigma(t) = \frac{1}{2} - \frac{t}{T};\ 0 < t < T \tag{19}$$

A proof is given in Appendix 6.2.

Asymptotic Limit Cycle Periodicity in σ

Efflorescent functions can be asymptotically periodic as a function of the expansion scaling variable σ.

Theorem 4.9. *Using the scalar periodicity matrix $\mathbf{Q} = T$, the one dimensional kernel*

$$g(t) = \Pi(t)$$

results in a efflorescent function periodic in σ with a period of $T_\sigma = 2T$.

Proof. The one dimensional Fourier series expression for the

[12]For $M = 5$ and $\vec{i} = [4\ 0\ 1]^T$, for example,

$$\left(\prod_{k=1}^{M} \frac{\partial^{i_k}}{\partial t_k^{i_k}} \right) g(\vec{t}) = \frac{\partial^5}{\partial t_1^4 \partial t_3} g(\vec{t}).$$

[13]Equal to one for positive argument and zero otherwise.

efflorescent function is

$$\zeta_\sigma(t) = \frac{\sigma}{T} \sum_{m \neq 0} \operatorname{sinc}\left(\frac{\sigma m}{T}\right) e^{i2\pi mt/T}. \qquad (20)$$

where

$$\operatorname{sinc}(u) := \frac{\sin(\pi u)}{\pi u}$$

is the one dimensional Fourier transform of $\Pi(t)$. For $m \neq 0$, the mth Fourier series coefficient is

$$\begin{aligned} c_\sigma[m] = \sigma G\left(\frac{\sigma m}{T}\right) &= \frac{\sigma}{T} \operatorname{sinc}\left(\frac{\sigma m}{T}\right) \\ &= \\ &= \frac{1}{T} \frac{\sin\left(\frac{\pi m \sigma}{T}\right)}{\pi m} \end{aligned}$$

Because of the sin term, $c_\sigma[m]$ is periodic with respect to σ with period $2T$.

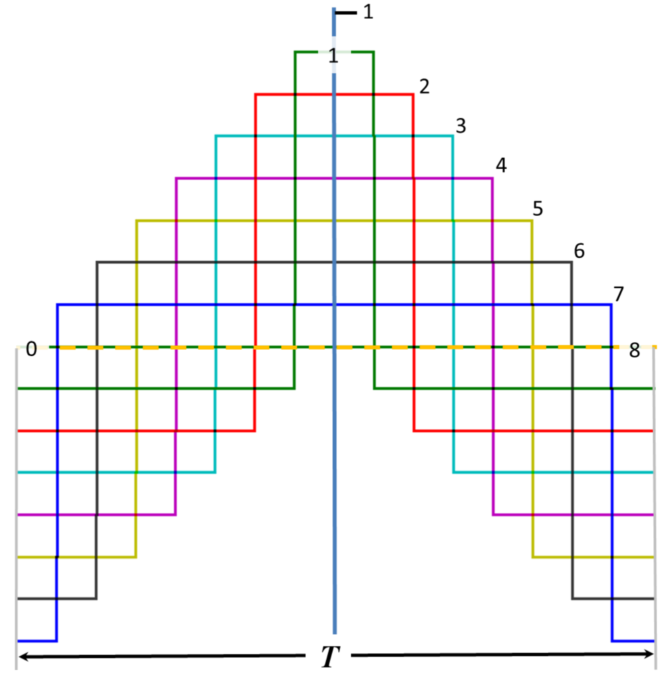
$$c_{\sigma+2T}[m] = c_\sigma[m]$$

Since all of the Fourier series coefficients in (20) are periodic with period $2T$, we conclude that

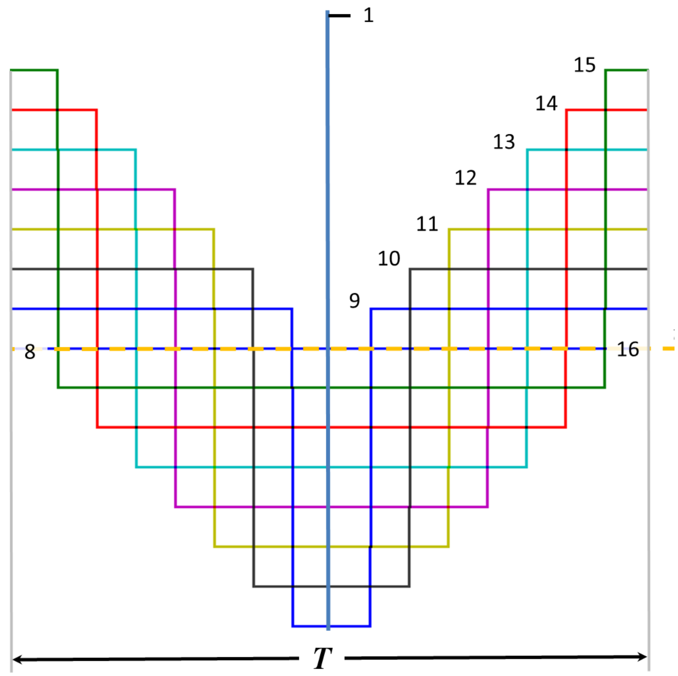
$$\zeta_{\sigma+2T}(t) = \zeta_\sigma(t).$$

and the efflorescent function oscillates as a function of σ. $\square$

The periodicity of $\zeta_\sigma(t)$ is illustrated in Figure 20. The function is bounded by $|\zeta_\sigma(t)| \leq 1$ and is plotted over a single period. The Figure begins with an all zero function marked with the number 0. σ increases a bit. For the function marked 1, there is now a short positive pulse and the remainder of the function is zero. Since $\langle \zeta_\sigma \rangle = 0$, all of the functions shown have zero area. The pulse at the origin begins to spread as σ increases as is seen in the functions marked 2 through 7. Then, at 8, the function returns to being identically zero. The second phase is shown in Figure 20b. We begin with the zero function marked 8 in Figure 20a which is also marked 8 in Figure 20b. σ increases a bit. The function marked 9 is a short negative pulse. As σ increases, the negative pulse widens as is seen in the functions marked 9 through 15. Function 16 is identically zero and is the same as the function marked 0 in Figure 20a. One period is complete and, as σ increases, the next identical period begins.



(a) The first part of the periodicity in σ.



(b) The second part.

Figure 20: The periodicity in σ of $\zeta_\sigma(t)$ for expanding rectangular kernels in (20). The period in σ is $2T$. The function $\zeta_\sigma(t)$ is also periodic in t with period T.

4.3 Efflorescent Examples

The most interesting of efflorescent functions are those that have no fixed asymptotic convergence properties. In two dimensions, such efflorescent functions continually bloom in a nonrepeating manner. The reader is encouraged to view the online videos (Nybal, 2014d; Nybal, 2014c; Nybal, 2014b; Nybal, 2014a) (especially the expanding pillbox cone and circular cone videos) to fully appreciate this emergence. Screen shots for expanding circle and cone shaped kernels are shown in Figures 3, 14 and 15. We now analyze properties of these two kernel types.

We find useful the radial measures

$$r = \|\vec{t}\| \text{ and } \rho = \|\vec{u}\|$$

where

$$\|\vec{t}\| = \sqrt{t_1^2 + t_2^2}. \tag{21}$$

When the variable $r = r(t_1, t_2)$ is used in a two dimensional expression, we assume it to be a two dimensional function of t_1 and t_2 as expressed here. Doing so avoids writing out the square root expression in (21) at each usage. The variable $\rho = \rho(u_1, u_2)$ can similarly be interpreted as a function of u_1 and u_2.

Expanding Pillbox Circle Example

The expanding circles example is a special case of (8) for $M = 2$ dimensions where the kernel $g(\vec{t})$ is one inside a circle of unit radius and is otherwise zero.[14]

$$g(\vec{t}) = \Pi\left(\frac{r}{2}\right). \tag{22}$$

Then $G(\vec{0})$ is simply the area of a unit radius circle. From the signal integral property in (2).

$$G(\vec{0}) = \int_{\vec{t}} \Pi\left(\frac{r}{2}\right) d\vec{t} = \pi.$$

For expanding circles, the expanding kernel function is

$$x_\sigma\left(\vec{t}\right) = \sum_{\vec{m}} \Pi\left(\frac{\|\vec{t} - \mathbf{Q}\vec{m}\|}{2\sigma}\right)$$

so that $g\left(\frac{\vec{t}-\mathbf{Q}\vec{m}}{\sigma}\right)$ in the $\vec{m}$th tile is a circle of radius σ centered at $\mathbf{Q}\vec{m}$.

The 2D Fourier transform of a unit radius circle in (22) is (Marks II, 2009)

$$G\left(\vec{u}\right) = \frac{J_1(2\pi\rho)}{2\rho}$$

where $J_1(\cdot)$ is a first order Bessel function of the first kind. Asymptotically (Abramowitz and Stegun, 1972)

$$\frac{J_1(2\pi\rho)}{2\rho} \underset{\rho\to\infty}{\longrightarrow} \frac{\rho^{-3/2}}{2\pi} \cos\left(2\pi\rho - \frac{3}{4}\pi\right) \tag{23}$$

[14] Specifically, from the definition of $\Pi(\cdot)$ in (17), $g(\vec{t}) = \Pi(r/2)$ is one for $\frac{r}{2} < \frac{1}{2}$. This is equivalent to $r = \|\vec{t}\| = \sqrt{t_1^2 + t_2^2} < 1$ which defines a circle of unit radius.

so that, for $\vec{m} \neq \vec{0}$,

$$\begin{aligned} \sigma^2 G(\sigma\mathbf{P}\vec{m}) &= \sigma^2 \frac{J_1(2\pi\sigma\|\mathbf{P}\vec{m}\|)}{2\sigma\|\mathbf{P}\vec{m}\|} \\ &\underset{\sigma\to\infty}{\longrightarrow} \frac{1}{2\pi}\sqrt{\frac{\sigma}{\|\mathbf{P}\vec{m}\|^3}} \cos\left(2\pi\sigma\|\mathbf{P}\vec{m}\| - \frac{3}{4}\pi\right) \end{aligned}$$

The sufficient condition of Theorem 4.4 for convergence of the efflorescent function to zero is therefore not met.

Pillbox Expansion on a Hexagonal Grid We have yet to specify a periodicity for the expanding circles. Assume the circle centers are spaced hexagonally in accordance with the periodicity matrix in (3). From (4) we see that

$$|\det \mathbf{P}| = \frac{1}{|\det \mathbf{Q}|} = \frac{2}{\sqrt{3}R^2}. \tag{24}$$

The expanding kernel function's mean, from (10), is therefore

$$\langle x_\sigma \rangle = \sigma^2 \left(\frac{2}{\sqrt{3}R^2}\right)\pi = \frac{2\pi\sigma^2}{\sqrt{3}R^2}$$

Screen shots for the efflorescent function is shown in Figure 3 from the video available online (Nybal, 2014d).

Expanding Circular Cones

Set the two dimensional kernel to a circular cone of unit height.

$$g\left(\vec{t}\right) = (1 - r)\Pi\left(\frac{r}{2}\right)$$

For circularly symmetric functions, the 2D Fourier transform becomes the Hankel transform (Marks II, 2009) so that (WolframAlpha.com, 2020) becomes

$$\begin{aligned} G\left(\vec{u}\right) &= 2\pi \int_0^1 r(1-r)J_0(2\pi r\rho)dr \\ &= \frac{H_0(2\pi\rho)J_1(2\pi\rho) - H_1(2\pi\rho)J_0(2\pi\rho)}{4\rho^2}. \end{aligned} \tag{25}$$

where $H_n(\cdot)$ are *Struve functions* (Weisstein, 2020) and $J_n(\cdot)$ are Bessel functions of the first kind. In Appendix 6.3, we show that

$$\sigma^2 G(\sigma\mathbf{P}\vec{m}) \underset{\sigma\to\infty}{\longrightarrow} -\frac{2}{\pi^2}\sqrt{\frac{\sigma^3}{\|\mathbf{P}\vec{m}\|}} \cos\left(2\pi\sigma\|\mathbf{P}\vec{m}\| - \frac{\pi}{4}\right). \tag{26}$$

As is the case with the expanding circles, the condition of Theorem 4.4 for convergence to the mean for the expanding cones is therefore not met. For any fixed $\mathbf{P}$ and $\vec{m} \neq \vec{0}$, the limit does not approach zero as σ increases without bound.

Circular Cone Expansion on a Hexagonal Grid Assume the cone centers are spaced hexagonally in accordance with the periodicity matrix in (3). Thus we can use (24). Since $J_0(0) = 1$, the volume of a cone with a unit circle base and unit height is, from (25),

$$G(\vec{0}) = 2\pi \int_0^1 r(1-r)dr = \frac{\pi}{3}.$$

The expanding kernel function's mean, from (10), is then

$$\langle x_\sigma \rangle = \sigma^2 \left(\frac{2}{\sqrt{3}R^2} \right) \frac{\pi}{3} = \frac{2\sqrt{3}\pi\sigma^2}{R^2}.$$

As was the case for the circular pillbox, the limit does not approach zero as σ increases without bound. Screen shots for the cone's efflorescent function are shown in Figures 14 and 15 from the video available online (Nybal, 2014c).

5 Conclusions

We have introduced the idea of periodic expanding kernel and efflorescent functions and have shown they can display widely variant behaviors dependent on the kernel and the underlying periodicity. Examples are given of efflorescent functions that converge to zero, converge to a nonconstant fixed point and oscillate. When the efflorescent functions fluctuate without repeating, patterns reminiscent of continual blooming can emerge. Special occurrences for a circular pillbox kefor of the three petal geometry representing Christianity's trinity and the flower-of-life. All emergent patterns are periodic and can be used for artful tiling.

6 Appendices

6.1 Proof of Theorem 4.5: Convergence to the mean

The derivative theorem of Fourier analysis indicates

$$\left(\prod_{k=1}^{M} \frac{\partial^{i_k}}{\partial t_k^{i_k}} \right) g\left(\vec{t}\right)$$

has a Fourier transform of

$$\left(\prod_{k=1}^{M} (j2\pi u_k)^{i_k} \right) G\left(\vec{u}\right)$$

Thus

$$G\left(\vec{u}\right) = \frac{\int_{\vec{t}} \left[\left(\prod_{k=1}^{M} \frac{\partial^{i_k}}{\partial t_k^{i_k}} \right) g\left(\vec{t}\right) \right] e^{-i2\pi \vec{t}^T \vec{u}} d\vec{t}}{\prod_{k=1}^{M} (j2\pi u_k)^{i_k}}.$$

and

$$\begin{aligned} \left|G\left(\vec{u}\right)\right| &= \frac{\left| \int_{\vec{t}} \left[\left(\prod_{k=1}^{M} \frac{\partial^{i_k}}{\partial t_k^{i_k}} \right) g\left(\vec{t}\right) \right] e^{-j2\pi \vec{t}^T \vec{u}} d\vec{t} \right|}{(2\pi)^{\tilde{M}} \prod_{k=1}^{M} |u|_k^{i_k}} \\ &= \\ &\leq \frac{\int_{\vec{t}} \left| \left(\prod_{k=1}^{M} \frac{\partial^{i_k}}{\partial t_k^{i_k}} \right) g\left(\vec{t}\right) \right| d\vec{t}}{(2\pi)^{\tilde{M}} \prod_{k=1}^{M} |u|_k^{i_k}} \\ &= \\ &= \frac{A}{(2\pi)^{\tilde{M}} \prod_{k=1}^{M} |u|_k^{i_k}} \end{aligned}$$

where

$$\tilde{M} = \sum_{k=1}^{M} i_k. \tag{27}$$

Continuing

$$\sigma^M \left|G(\sigma\vec{u})\right| \leq \frac{A\sigma^M}{(2\pi\sigma)^{\tilde{M}} \prod_{k=1}^{M} |u|_k^{i_k}}$$

and

$$\sigma^M \left|G(\sigma\mathbf{P}\vec{m})\right| \leq \frac{A\sigma^{M-\tilde{M}}}{(2\pi)^{\tilde{M}} \prod_{k=1}^{M} |(\mathbf{P})_k\vec{m}|^{i_k}}.$$

With all other parameters fixed, this expression tends to zero for increasing σ when

$$M - \tilde{M} < 0$$

or, using (27),

$$\sum_{k=1}^{M} i_k > M$$

6.2 Proof of Theorem 4.8: Convergence to a fixed periodic function

Applying the exponential kernel in (18) to (8) for scalar $\mathbf{Q} = T$ gives

$$\begin{aligned} x_\sigma(t) &= \sum_{m=-\infty}^{\infty} \exp\left(-\frac{t-mT}{\sigma} \right) \mu(t-mT) \\ &= e^{-t/\sigma} \sum_{m=-\infty}^{\infty} e^{mT/\sigma} \mu(t-mT) \end{aligned}$$

Over the period $0 \le t < T$,

$$x_\sigma(t) = e^{-t/\sigma} \sum_{m=-\infty}^{0} e^{mT/\sigma}.$$

Using a geometric series

$$x_\sigma(t) = \frac{e^{-t/\sigma}}{1 - e^{-T/\sigma}}$$

and

$$\zeta_\sigma(t) = x_\sigma(t) - \frac{\sigma}{T}.$$

Express the exponentials as a truncated Taylor series.

$$\zeta_\sigma(t) = \frac{1 - \frac{t}{\sigma} + \frac{t^2}{2\sigma^2}}{\frac{T}{\sigma} - \frac{T^2}{2\sigma^2}} - \frac{\sigma}{T}.$$

After some manipulation

$$\zeta_\sigma(t) \underset{\sigma\to\infty}{\longrightarrow} \frac{\sigma\left(t - \frac{T}{2}\right) + \frac{t^2}{2}}{T\left(\sigma - \frac{T}{2}\right)} \underset{\sigma\to\infty}{\longrightarrow} \frac{1}{2} - \frac{t}{T}$$

which is the desired result in (19).

6.3 Proof of (26): Cone convergence

The Struve functions used in (26) can be defined by their Taylor series (Weisstein, 2020)

$$H_0(z) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{[(2k+1)!!]^2}$$

and

$$H_1(z) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k}}{(2k-1)!!(2k+1)!!}.$$

They have the following asymptotic behavior (Wolfram Research, 2020b)

$$H_0(z) \underset{|z|\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4}\right)\left(1 + O\left(\frac{1}{z^2}\right)\right)$$

and

$$H_1(z) \underset{|z|\to\infty}{\longrightarrow} \frac{2}{\pi}\left(1 + O\left(\frac{1}{z^2}\right)\right).$$

Likewise, the Bessel function has the asymptotic behavior (Wolfram Research, 2020a)

$$J_0(z) \underset{|z|\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right)\left(1 + O\left(\frac{1}{z^2}\right)\right)$$

and

$$J_1(z) \underset{|z|\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{3\pi}{4}\right)\left(1 + O\left(\frac{1}{z^2}\right)\right)$$

In light of these behaviors, inspection of (26) reveals the $H_1(z)J_0(z)$ term asymptotically dominates the $H_0(z)J_1(z)$ term and

$$H_1(2\pi\rho)J_0(2\pi\rho) \underset{\rho\to\infty}{\longrightarrow} \frac{2}{\pi^2}\rho^{-\frac{1}{2}} \cos\left(2\pi\rho - \frac{\pi}{4}\right).$$

Thus

$$\sigma^2 G(\sigma\vec{u}) \underset{\sigma\to\infty}{\longrightarrow} -\frac{2}{\pi^2}\sqrt{\frac{\sigma^3}{\rho}} \cos\left(2\pi\sigma\rho - \frac{\pi}{4}\right)$$

from which (26) follows.

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Deciding a Bitstring of 1s is Non-Random is Impossible in General

Eric Holloway

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Theorem 1 (Chaitin's incompleteness theorem)**.** For every axiomatic proof system T that can be encoded in at least N_T bits, there is a constant c such that the Kolmogorov complexity of a bitstring b cannot be proven to be larger than c.

We can phrase this in terms of an algorithm $A_T(b, n, t)$ that checks every possible proof in T. When applied to a bitstring b and a value n, $A_T(b, n, t)$ outputs a proof in T that $K(b) > n$, or fails after t steps and outputs a null, $\{\}$,

$$\exists c : \forall b, n \geq c, t, A_T(b, n, t) = \{\}.$$

Proof. This is apparent by assuming the opposite,

$$\forall n, \exists b, t, A_T(b, n, t) \neq \{\}. \tag{1}$$

We construct an algorithm B_T^n which, for a given n, performs a breadth first search through all bitstrings b and step amounts t with $A_T(b, n, t)$ until it finds a proof that $K(b) > n$. B_T^n then outputs b. By assumption in Equation 1, B_T^n will always halt.

The Kolmogorov complexity of B_T^n is $K(B_T^n) \leq K(A_T) + \log n + \beta$, where β is a constant for the overhead in B_T^n. The value of β is independent of n, so will not vary as n changes.

We then set $n = c$ such that $K(B_T^c) \leq K(A_T) + \log n + \beta < c$. Since B_T^c will halt, it will output a b such that $K(b) > c$. However, $K(B_T^c) < c$, resulting in a contradiction. Thus, for all $n \geq c$ our assumption in Equation 1 is false.

Therefore, Theorem 1 is true. □

Theorem 2 (Can prove non-randomness)**.** We define a function $U_T(b, n, t)$ which outputs proofs in T of the form $K(b) < \ell(b)$, which are proofs of non-randomness. It is parameterized in the same way as $A_T(b, n, t)$.

For all lengths c of bitstrings b, there is a T that can prove at least one bitstring of length $\ell(b) = c$ is non-random,

$$\exists b \forall c, n, t, U_T(b, n, t) \neq \{\} \wedge \ell(b) = c.$$

Proof. We begin by assuming the contrary

$$\exists c' \forall c > c', n, t, U_T(b, n, t) = \{\} \wedge \ell(b) = c. \tag{2}$$

The following axiomatic system T_{1s} is a falsification of Equation 2.

Axioms of T_{1s} are:

1. $K(\{1\}^{20}) < 20$.
2. If $K(b) < \ell(b)$, then $K(b1) < \ell(b1)$.
3. If $K(b) < \ell(b)$, then b is non-random.

For any bitstring of 1s b_{1s} where $\ell(b_{1s}) \geq 20$, T_{1s} can prove the bitstring is non-random. It does this by incrementally building a bitstring of 1s until the input is matched. The axioms of T_{1s} are true and all proofs are by induction, so all proofs are true. Equation 2 is contradicted. □

Theorem 3 (Cannot generally prove non-randomness)**.** There is no axiomatic system that can decide the non-randomness of every non-random bitstring.

Proof. While a dovetailing algorithm can output proofs of non-randomness for every non-random bitstring, there is no decision procedure that can decide whether the dovetailing algorithm will halt. If there were, then this decision procedure can enumerate all random bitstrings, contradicting Theorem 1. □

Even though a human can trivially decide an arbitrarily long bitstring of 1s is not random, Theorem 3 shows is an impossible task for a generalized algorithm. Only a specific algorithm, such as exemplified in Theorem 2, can do so.

This conclusion is a bit counter-intuitive, since it means that without domain knowledge, an algorithm given an extremely long sequence of 1s would be unsure whether the sequence is completely random. When asked to predict the next digit, the algorithm can only give an equal weighting to 0 and 1.

Proving the Derivative of $\sin(x)$ Using the Pythagorean Theorem and the Unit Circle

Jonathan Bartlett

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The derivative of $\sin(x)$ (where x is measured in radians) is given in standard calculus as $\cos(x)$. The proof for this is usually based on a limit: $\lim\limits_{q \to 0} \frac{\sin(q)}{q} = 1$. The proof, put simply, is:

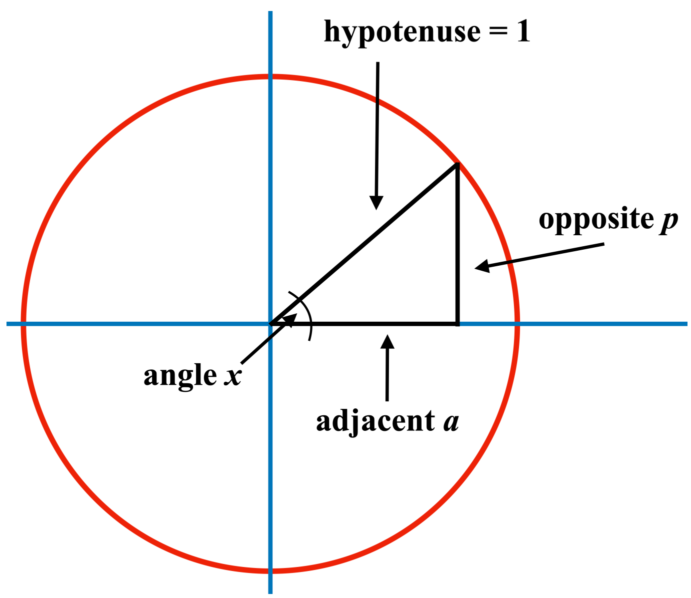
$$y = \sin(x) \tag{1}$$
$$y + \mathrm{d}y = \sin(x + \mathrm{d}x) \tag{2}$$
$$\mathrm{d}y = \sin(x + \mathrm{d}x) - \sin(x) \tag{3}$$
$$\mathrm{d}y = \sin(x)\cos(\mathrm{d}x) + \cos(x)\sin(\mathrm{d}x) - \sin(x) \tag{4}$$
$$\mathrm{d}y = \sin(x) + \cos(x)\sin(\mathrm{d}x) - \sin(x) \tag{5}$$
$$\mathrm{d}y = \cos(x)\sin(\mathrm{d}x) \tag{6}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x)\frac{\sin(\mathrm{d}x)}{\mathrm{d}x} \tag{7}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x) \tag{8}$$

While there is nothing wrong with the proof per se, I have always found it unsatisfying, utilizing trigonometry identities few students remember. Additionally, it is usually accompanied with an explanation of the limit of $\frac{\sin x}{x}$ that is hard for students to decipher. Therefore, this paper endeavors to provide a more straightforward proof based on more basic mathematical assertions, founded on the Pythagorean theorem and the unit circle. It doesn't remove the given limit in its entirety, but rather gives more straightforward, calculus-oriented reasoning for doing a similar operation. It is debatable how much different it is *in kind* from the standard proof, but in any case I think it is a more straightforward, interesting, and instructive way of looking at it for students. It shows (a) the power of calculus, (b) the power of differential thinking, and (c) how discoveries can be made from basic principles.

Basic Assumptions

This proof will be analyzing triangles drawn on the unit circle. On a unit circle, the hypotenuse will always be 1. Figure 1 shows the general setup. x will be the angle measured in radians, a will be the adjacent, and p will be the opposite.

Figure 1: A Triangle Inscribed Onto a Unit Circle



The Pythagorean theorem gives the following:

$$a^2 + p^2 = 1 \tag{9}$$
$$p^2 = 1 - a^2 \tag{10}$$
$$a^2 = 1 - p^2 \tag{11}$$
$$\tag{12}$$

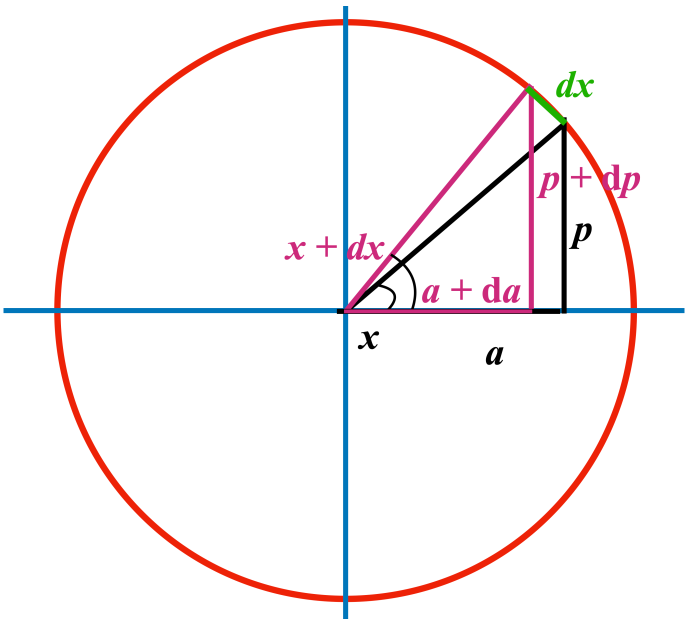
Since the hypotenuse is 1, $\sin(x) = p$ and $\cos(x) = a$. The derivative of $\sin(x)$ with respect to x, therefore, will be $\frac{\mathrm{d}p}{\mathrm{d}x}$. Therefore, the proof will be successful if it can demonstrate the following equivalency:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = a \tag{13}$$

Differential Analysis

Taking Figure 1 and budging the angle by $\mathrm{d}x$ yields the picture shown in Figure 2.

Figure 2: Changes in the Triangle Based on $\mathrm{d}x$



A few important notes on Figure 2:

1. All changes are being expressed as *adding* differentials, even if the differential itself is negative. This is why $a+\mathrm{d}a$ in the graph is shorter than a on its own.
2. Since this is the unit circle, the angle change *is identical* to the circumference change (since the radius is 1, then the circumference is 2π, the number of radians in a circle).
3. Since the change is infinitesimal, and this is a smooth and continuous figure, then the change on the differential is *linear*. In other words, the picture is zoomed in enough that the arc joining the two triangles can be treated as if it were a straight line.[1]

Because of this last point, the length of $\mathrm{d}x$ can be determined using the distance formula, where the horizontal and vertical changes are simply given by $\mathrm{d}a$ and $\mathrm{d}p$:

$$\mathrm{d}x = \sqrt{\mathrm{d}p^2 + \mathrm{d}a^2} \tag{14}$$

Finally, the differential of (9) can be taken to come up with:

$$a^2 + p^2 = 1$$

$$2a\,\mathrm{d}a + 2p\,\mathrm{d}p = 0 \tag{15}$$

$$a\,\mathrm{d}a + p\,\mathrm{d}p = 0 \tag{16}$$

$$a\,\mathrm{d}a = -p\,\mathrm{d}p \tag{17}$$

$$\mathrm{d}a = -\frac{p}{a}\,\mathrm{d}p \tag{18}$$

Making the Proof

Starting with (14), substitutions and simplifications can be made as follows:

$$\mathrm{d}x = \sqrt{\mathrm{d}p^2 + \mathrm{d}a^2} \tag{19}$$

$$= \sqrt{\mathrm{d}p^2 + \left(-\frac{p}{a}\,\mathrm{d}p\right)^2} \tag{20}$$

$$= \sqrt{\mathrm{d}p^2 + \frac{p^2}{a^2}\,\mathrm{d}p^2} \tag{21}$$

$$= \sqrt{\mathrm{d}p^2 + \frac{1-a^2}{a^2}\,\mathrm{d}p^2} \tag{22}$$

$$= \sqrt{\mathrm{d}p^2 + \frac{\mathrm{d}p^2}{a^2} - \mathrm{d}p^2} \tag{23}$$

$$= \sqrt{\frac{\mathrm{d}p^2}{a^2}} \tag{24}$$

$$\mathrm{d}x = \frac{\mathrm{d}p}{a} \tag{25}$$

Note that (25) could also have been negative. Inspection of Figure 2 shows that $\mathrm{d}p$ will always have the same sign as a (increasing until a is zero, then decreasing while a is negative). Therefore, choosing the positive square root is the valid choice.

As stated at the beginning, the goal is to figure out an alternative reading of $\frac{\mathrm{d}p}{\mathrm{d}x}$. Using (25), this can be simplified as follows:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}p}{\frac{\mathrm{d}p}{a}} = \frac{\mathrm{d}p}{1}\frac{a}{\mathrm{d}p} = a \tag{26}$$

As shown in (13), this proves that the derivative of $\sin(x)$ is indeed $\cos(x)$.

[1] Note that this is basically equivalent to the limit $\lim_{q\to 0}\frac{\sin(q)}{q}$, but stated in a more straightforward way that is repeatedly in calculus thinking.

What is significant about this proof is that it relies entirely on the basics—the Pythagorean theorem, the unit circle, the definition of sine and cosine, the definition of the radian measure of an angle, the distance formula, and the power rule.

A Response to Clunn's Axioms of Morality

J R Miller

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This article offers a brief critique of Clunn's foundationalism which grounds moral decision making in what he calls the three fundamental axioms of existence, consciousness, and identity (Clunn, 2019). It shows how his commitment to neo-Platonism, or possibly pantheism, creates at least three incoherencies wherein *a priori* is *a posteriori*, individuality is an illusion, and objective morality is subjective. For Clunn's moral philosophy to offer practical value, these internal conflicts must be resolved.

Introduction

In his article, *Axioms of Morality*, Clunn argues that morality is an *a priori* truth that is objectively known to every person. He believes that the fundamental axioms of existence, consciousness and identity make life itself the ultimate objective standard for each person's subjective moral choices. Therefore, as a general rule, any moral choice which benefits life in general is a moral good. Any moral choice which hurts life in general is a moral evil. Clunn rejects selfishness and utilitarianism as viable methods for choosing what is good. Instead, he argues that our individual choices must be guided by what he considers the four cardinal virtues defined by history: justice, prudence, temperance and courage.

Finding an objective ground for moral good is a daunting task for any philosopher. And while Clunn's three axioms are important, the grounding for his overarching moral philosophy is problematic. For Clunn, every person shares in the same *a priori* universal consciousness which is a nonreductive emergent property of the biological structures that define humanness. If it is true that existence, consciousness and identity exist *a priori* to human life in some form of neo-Platonic realm—or possibly in a pantheistic universe—at least three internal conflicts arise.

Conflict #1: *A Priori* is *A Posteriori*

Clunn leans heavily on Ayn Rand for defining his axioms but departs from Rand who taught that consciousness and morality are *a posteriori*. This distinction is critical for Clunn as he hopes to sustain his commitment to both objective morality and free will. He writes, "At the end of the day, morality is about free will, choices, and decisions. These things all exist within our consciousness (45)." Here is where the incoherence first manifests. By definition, *a priori* means that morality must exist independent of any person's experience. Yet, Clunn also presumes that morality exists through the exercise of one's free will. Given these claims, morality must also be *a posteriori* because it depends upon how each individual person exercises their free will. Clunn's presupposition of *a priori* morality may be preserved if he assumes free will is also an expression of the *a priori* universal consciousness. However, this assumption leads to a second conflict for how Clunn defines identity.

Conflict #2: Individuality is an Illusion

For Clunn, consciousness is not a property of personhood, but an emergent property of the physical realm that existed before any individual. That is to say, all humans share in the one nonreductive *a priori* universal consciousness. At the same time, Clunn argues that the term "I" is an expression of rational thought which establishes one's specific identity within the universal consciousnesses. But even if "I" establishes my personal existence, it remains an existence only within the larger axiom of existence. It seems to follow from Clunn's own definitions that the perception of individuality, and by extension free choice, is only an illusion. This creates at least one significant internal conflicts for Clunn's axioms.

Clunn argues that the consequences of our decisions are experienced only within the realm of personal consciousness, which no other person can observe. In contrast, Clunn says that we can observe existence. Yet, for Clunn, morality does not manifest in the axiom of existence. However, if each person's consciousness is a shared *a priori* reality, how is it beyond my powers of observation? If I can have awareness of my own consciousness, and that consciousness is tied to the universal, then by definition I must also have access to understanding the consciousness of others because they too are tied to the same universal axiom. Even more, if consciousness is an emergent property of existence, how does it remain independent of existence as it relates to morality? This incoherence leads to the final issue addressed herein regarding Clunn's claim that objective morality exists.

Conflict #3: Objective Morality is Subjective

According to Clunn, the goal of his work is to establish a simple way for people to make moral choices that do not rely on some deeply esoteric philosophy. While this goal is laudable, Clunn's claim to objective moral reality remains frustratingly subjective. Even if we grant that consciousness is a fundamental axiom and that morality exists in some neo-Platonic form, this says nothing about how we should evince this unobservable moral realm. How does Clunn's foundationalism solve the is/ought problem? Clunn concedes that even given his belief in an objective moral realm, "there is not a physical imperative to adhere to it (45)." In the end, Clunn's morality seems like a philosophically complex version of the equally impractical moral trope, 'follow your heart.'

Clunn's claim to objective morality ends in a confusing subjectivity. He writes, "I don't prescribe here any specific actions that a human should take to be moral or immoral. Specific actions can and should be subjective to each individual (44–45)." Now to suggest that morality is objective, but all individual actions are subjective and beyond judgement from others only begs the question about what it means to value life? Clunn is aware that his dualist morality leads to moral confusion and leaves open the argument that 'might makes right.' To evade this looming problem, he asserts that consciousness is an emergent function of existence, but he fails to show how this claim logically undermines the 'might makes right' philosophy. Even worse, Clunn admits that taking life is unjust, but believes his own willingness to take a human life is virtuous. He writes, "I have no qualms, nor does my morality, with taking a life if that life is deciding to destroy other lives (45)." But if, as he claims, there is no basis to judge the conscientious free moral choice of another person, on what foundation—outside of his own subjective opinion—does Clunn decide it is right to end the life of another person? Maybe the person Clunn chose to kill had a reason within their own mind that justified their choice to take a life. Maybe they killed in the hope that their act would serve the higher virtue of protecting lives. On what ground does Clunn judge their act as an evil? And what if my subjective morality tells me that Clunn's choice to kill is wrong? What would keep me from continuing this same cycle of violence? How can Clunn claim to preserve the general principle of life, without protecting the individual life of each person?

Conclusion

This article offered a brief critique of Clunn's foundationalism which grounds moral decision making in what he calls the three fundamental axioms of existence, consciousness, and identity. It showed how his commitment to neo-Platonism, or possibly pantheism, creates at least three incoherencies wherein *a priori* is *a posteriori*, individuality is an illusion, and objective morality is subjective. For Clunn's moral philosophy to offer practical value, these internal conflicts must be resolved.

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Is Information Content a Single, Static Quantity?

Jorge Fernandez

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Information is instinctively and commonly regarded as a single, static entity. For example, upon learning that 'her name is Susan' we would say that we have 'acquired information'. Taken that way, an instance of information naturally leads to a single, static quantity of information. Thus, if asked how much information is contained in learning that a flip of a fair coin yielded a 'heads', the typical answer is '1 bit of information' ($-\log_2\left(\frac{1}{2}\right) = 1$ bit).

Seasoned information theorists are, of course, aware that there is more to it than that. Yet even those individuals usually operate within a paradigm that is most often not as comprehensive as it needs to be to address the full information picture. Thus, '1 bit' merely indicates the number of possible states ($2^1 = 2$ states). Alone, this number is indifferent regarding deeper, significant aspects of information such as *meaning*.

There is an attribute about information that is even more fundamental than those considerations. I am specifically referring to the fact that before information may be measured it must first manifest as a specific kind of information, and that manifestation always occurs within a fixed context. If any critical element of the context is changed, the information that is

News

Blyth Institute Director Publishes Book on Introductory Electronics

The director of the Blyth Institute, Jonathan Bartlett, recently published an introductory electronics book titled *Electronics for Beginners: A Practical Introduction to Schematics, Circuits, and Microcontrollers*, published by Apress. The book is usable either as a textbook or as a self-study guide, containing numerous examples as well as exercises at the end of each chapter. It hits a midway point between typical hobbyist and professional electronics books, providing enough technical details for hobbyists to really understand what is going on, but not requiring a heavy mathematics background.

Blyth Institute Members Interviewed on the MindMatters Podcast

Blyth Institute fellow Eric Holloway and director Jonathan Bartlett were guests on the MindMatters podcast. The MindMatters podcast provides news and commentary on technology with a focus on artificial intelligence. In this series of episodes, Holloway and Bartlett looked at both the promise and failings of artificial intelligence. The series covered both the top ten uses and advances of AI, as well as the "dirty dozen" misapplications, overhypes, and failings of the same.

The MindMatters Podcast can be found at `https://mindmatters.ai/podcast/`.

Fine Tuning Picks Up Momentum in Biology

The concept of "fine tuning" has long been a standard part of physics and cosmology. It first made its debut in 1961 with Robert Dicke pointing out that certain forces in physics have to be tuned within very narrow parameters for life to exist anywhere (Dicke, 1961). To the present day, the concept of fine tuning is a source of active research and discussion in the physics community.

Recently, however, the concept of fine tuning has started to emerge in many aspects of biology as well. The discussions around fine tuning began with a focus on the genetic code and the codon table. Philip and Freeland (2011) noted that, while there are many biologically possible codon tables, the one that actually occurs in biology is comparatively highly tuned to match the spectrum of available amino acids. Others, such as Castro-Chavez (2012), have pointed out that the codon table has a lot of internal, logically-consistent structure to it. The discussion of various ways that the genetic code has been optimized continues to produce new insights (José and Zamudio, 2020).

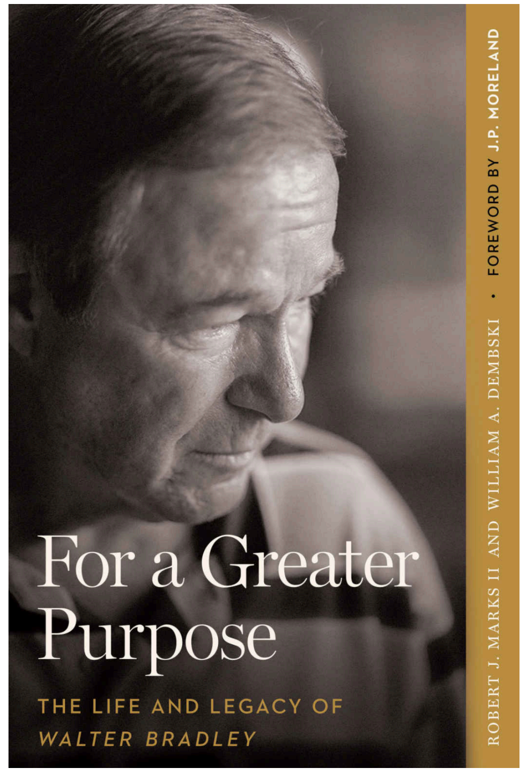
Other aspects of fine tuning have been discovered as well. A lecture by William Bialek gives an overview of many biological systems which operate at or very near the highest resolution allowed by physics (Bialek, 2015). This included discoveries such as the ability for bacteria to count individual molecules on their surfaces and the ability of a retinal cell to respond to a single photon.

The growth of this phenomena has caused some biologists to start approaching the question of fine tuning with more rigor. Recently, the Journal of Theoretical Biology published a paper which reviewed various statistical approaches of analyzing fine tuning quantitatively (Thorvaldsen and Hössjer, n.d.), and notes that, while there are many open questions, the field has already gone in many promising directions.

New Book on Walter Bradley's Life and Legacy

Walter Bradley has had a significant impact on members of The Blyth Institute. He was the plenary speaker at our first conference, discussing both the fine tuning of the laws of physics as well as his work in using technology to help the poor in rural economies.[1]

In August of this year, Erasmus Press published a book about his life titled *For a Greater Purpose: The Life and Legacy of Walter Bradley*, written by Robert J. Marks and William A. Dembski. This book covers not only his academic and engineering achievements, but also his personal connections with students, faculties, and other organizations, and the impact that his life has had on all of these.



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[1] His talk, "Unique Ways for Engineers to Bring Healing," can be found at https://youtu.be/X92BDBku6g4.