Applying Calculus

Applying Calculus
Lessons We Can Learn about Money, Physics, Biology, and Other Subjects from Mathematics' Greatest Achievement

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by
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Contents

Chapter 1

Introduction

1.1 PREVIEW

This is a preview of the book. If you receive a preview of this book, help me out! Send any suggestions to jonathan@bartlettpublishing.com. I'm planning on self-publishing this, but if you know publishers who might be interested, hook me up or send me an email. Also, if you think there is a good calculus equation that first-year students should be aware of, send me a suggestion!

1.2 On to the Book

Many calculus courses are taught in a way that focuses on the mathematics of calculus. This is all well and good—you need to know the mechanics of the mathematics in order to really understand calculus. However, just learning calculus often leaves people wondering "why bother?"

It turns out that, since calculus was introduced, it has revolutionized our understanding of numerous different fields of study. The goal of this book is to give the reader a taste for what even introductory calculus brings to the understanding of our world in several diverse fields of study.

This book is a tour, not a systematic guide. The goal is to help the reader think in calculus terms—how to see what other people saw when they looked at the world and said, "calculus will solve it."

Many people think that equations fall out of the sky, are written on tablets of stone, or are simply conjured out of a deep magic. In reality, equations come from thinking about a situation, and imagining how various mathematical tools may be applied.

Imagine that you have just purchased a couch for your house. After that comes several steps—you have to figure out how you are going to transport it to your house, how you are going to get it through your front door, how you are to arrange your room so it will fit, and how many friends you need to have over to help you move it.

This sort of imagining and stepping through possibilities is exactly the same kind of thinking that goes into building all the great equations of history. You know the tools you have—friends, cars, a cart, etc. You know what you are trying to do—move the couch into your house. The goal is to get the tools you are using aligned to accomplish your goal.

If you're not used to thinking about math in this way, that's okay. Most people have been trained to do math in a somewhat formulaic way—you have a formula that does something, and then you have a problem that you match to the formula, and then you calculate a result. My goal is to help you to go beyond just using formulas, and learn how to be an inventor of formulas. My goal is to take you behind the scenes to how we think about mathematics and its application to the real world.

If you are a student, I hope that this book ignites your passion for mathematics and for what it can do for you. I hope that it helps you to see the world with new eyes.

If you are a teacher, I hope that this book gives you good examples to help your students. Calculus can sometimes seem dry and abstract, so hopefully this book can give you lots of examples that you can use to liven your discussions and show your students the power of calculus in the real world.

1.3 Common Conventions

FIXME: Dropping units to make things easier

FIXME: usually lowercase = variable and uppercase = constant, but important exceptions, especially the 0 subscript for initial conditions

FIXME: Leave off $+C$ from definite integrals

FIXME: How we refer to equations

FIXME: physics is newtonian

Chapter 2

Basic Techniques Used in this Book

Part I

Money

Chapter 3

The Interest Rate Equation

Compared to other chapters, this chapter is fairly simple and straightforward, and covers a topic you probably already covered in calculus. This chapter is here for two purposes. The first is to get you started with an easy one—it makes a good warm-up exercise. The second is that there are other equations in this book that use the principles and ideas developed here, so it is good to stop and take a minute and think about the interest rate equation.

3.1 Basic Interest Rate Equation

The interest rate equation is one that is taught in most grade-school textbooks. It says that, for interest compounded continuously, the formula for your final amount is

$$a = a_0 e^{Rt} \tag{3.1}$$

where a is your final amount of money, a_0 (often referred to as P for "principal") is your starting amount of money, R is the interest rate, and t is the amount of time (where time is in the same units as R).

Using this formula, if you had a savings account which started with $1,000 and paid out at a rate of 1% per year and was compounded continuously, then the amount in your account after three years would be

$$\begin{aligned} a &= a_0 e^{Rt} \\ &= 1,000 e^{0.01 \cdot 3} \\ &= 1,000 e^{0.03} \\ &\approx 1,000 \cdot 1.030 && = 1,030 \end{aligned}$$

When I say "compounded continuously," what I mean is that, while the rate is 1% per year, as that money gets added to my account, I *also* earn money on the money that has been added already. Therefore, the actual rate winds up being more than 1% because I also earn money on the interest that I accrue during the year.

3.2 The Model Equation

So where does this equation come from?

There are actually a lot of ways to get to it. The most straightforward way, however, is to think about it in terms of a derivative.

Since a is the amount of money that I have in my account, and t is the amount of time that has elapsed, then we would write the rate of accumulation of money in dollars (not as a percent—yet), as $\frac{\mathrm{d}a}{\mathrm{d}t}$. What is the rate that the money is accumulating at? Well, we know it as a percentage—it's 1% (0.01) per year. But $\mathrm{d}a$ refers to the change in dollar amount *in dollars*, not as a percentage.

However, if you think about it, to convert the percentage to actual dollars, we just need to multiply the rate times the current amount of money. If our current amount of money is a, and the rate is R, then the rate in actual dollars is simply Ra. Therefore, we can write the equation that models our system as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = Ra \tag{3.2}$$

where R is a constant (the interest rate—1% or 0.01).

3.3 Converting to the Interest Rate Equation

Now, to get into an equation that involves values rather than derivatives, we need to integrate. If we divide both sides by a and multiple both sides by $\mathrm{d}t$ it will yield

$$\frac{\mathrm{d}a}{a} = R\,\mathrm{d}t. \tag{3.3}$$

Integrating yields:

$$\int \frac{\mathrm{d}a}{a} = \int R\,\mathrm{d}t \tag{3.4}$$

$$\ln(a) = Rt + C \tag{3.5}$$

$$\tag{3.6}$$

Since we want to solve for a (and not $\ln(a)$), we can exponentiate both sides and come up with the following:

$$e^{\ln(a)} = e^{Rt+C} \tag{3.7}$$

$$a = e^{Rt}e^{C} \tag{3.8}$$

This is very similar to (3.1), but not quite. While (3.1) has a_0, (3.8) has e^C. But what is a_0? It is merely the value of a at time $t = 0$.

So let's look and see what the value of the equation is for a_0.

$$\begin{aligned} a &= e^{Rt}e^{C} \\ a_0 &= e^{R\cdot 0}e^{C} \\ a_0 &= e^{0}e^{C} \\ a_0 &= e^{C} \end{aligned}$$

So, a_0 and e^C are actually the *same quantity*. Therefore, replacing e^C with a_0 in (3.8) will give us the equation we were looking for, (3.1).

3.4 Other Applications of the Equation

The equation in (3.1) isn't just used for calulating the size of your bank account. There are actually a number of different things that operate on this basis.

Chapter 4

The Kelly Criterion: How Much Should You Bet (or Invest)?

Let's say that you are wanting to bet on a horse race. You brought \$1,000 to the race track (this is your *stash* or *bankroll*). You have analyzed every horse in the race. You have found that there is a particular horse for which the track is giving even odds (it's paying out as much as you bet), but you are 99% confident will win. How much should you bet on this horse?

4.1 Thinking About Strategies

You might think that, with odds like that in your favor, you should bet your whole stash. However, even if you are correct on the odds, if you follow that strategy, it is a sure loser.

Why?

Let's say that you encounter that situation over and over again, and each time you bet all your money. Since you are only 99% confident, that means that 1% of the time, you will be wrong. What happens when you are wrong? You lose *all* your money. This means that, even when you are 99% assured of victory, you shouldn't bet everything.

So how much should you bet?

Well, we want to bet *something.* It's rare that we are in such a circumstance, so we want to take advantage of it.

So, we know that we want to bet more than zero and less than everything. So what amount of money or percentage of our cash will likely give us the most money in return?

Did you notice what word I used? I used "most." Recognizing key words like this are key to using calculus well. When someone says "most," what they mean is that they want to *maximize* something. In calculus, we find the maximum of something by taking the derivative and setting it to zero. That will tell us our minimum and maximum points.

However, to use calculus, we first need a formula! I always tell calculus students that the actual calculus is actually the easy part of calculus. The hard part is figuring out how to *model* something so that we can *use* calculus.

4.2 Modeling the Betting Process

What we are going to do is to think about how to model the betting process. We will use m to represent the final amount of money, and m_0 as being our starting amount of money. So let's model a single bet where we lose.

4.2.1 Modeling Losing

When you lose, you simply lose the amount of money that you bet. This would mean

$$m = m_0 - \text{bet amount}. \tag{4.1}$$

However, this winds up not being very useful, because you will probably want to vary the amount that you bet based on how much cash you have on-hand. That is, we will want to bet some percentage of our current amount of money, not a specific dollar amount.

So, let's say that we bet 30% (0.3) of our money and lose. We can express that as saying that we get to *keep* 70% (0.7) of our money. So we can write this as

$$m = m_0 \cdot 0.7 \tag{4.2}$$

However, that seems kind of janky. We want to express this in terms of the portion that we *bet.* Therefore, a better way of saying this is

$$m = m_0(1 - 0.3) \tag{4.3}$$

We can even go further, and use a stand-in for the portion of our bankroll that we will be betting. We will call this r, for bankRoll portion. Now, the equation becomes

$$m = m_0(1 - r) \tag{4.4}$$

What if we lose twice in a row? Well, if we bet the same percentage no matter what the size of our bankroll, we just multiply by that again.

$$m = m_0(1 - r)(1 - r) \tag{4.5}$$

4.2.2 Modeling Winning

Okay, that's the losing side. When betting at a track, the winning side is a little more complicated. That's because the track tells you how much it is going to pay you back for winning. If the track doesn't think a horse is likely to win, it will pay you back more. If the track thinks the horse is likely to win, it will pay you back less. Obviously, if you win, you will always get your original bet back (i.e., you won't lose money). The additional amount you get for winning is known as the "betting odds," and is usually expressed as a fraction. So, if you bet \$300 on a horse with $\frac{1}{2}$ odds (pronounced "one-to-two odds"), that means that, in addition to keeping your money, you will earn an additional half of your money back.

So, if I have \$1,000, and I bet 30% at $\frac{1}{2}$ odds, then I will receive and extra \$150 back. So, let's keep r as the portion of our bankroll, and add in B as our "betting odds." So, a winning bet looks like this:

$$m = m_0(1 + r \cdot B) \tag{4.6}$$

The 1 means that we will certainly maintain our money, and the $r \cdot B$ means that the amount we get back due to betting odds is based on the portion that we bet.

So, what happens if we win twice in a row? We can just perform the same multiplication again.

$$m = m_0(1 + rB)(1 + rB) \tag{4.7}$$

4.2.3 Interweaving Wins and Losses

Now, on any bet, if we do it over and over again, sometimes we will win, and sometimes we will lose. Let's say that we lose once, then win twice, then lose again, then win again. What will that look like? Well, we just need to chain together our formulas for winning and losing:

$$m = m_0 \overbrace{(1-r)}^{\text{lose}} \overbrace{(1+rB)}^{\text{win}} \overbrace{(1+rB)}^{\text{win}} \overbrace{(1-r)}^{\text{lose}} \overbrace{(1+rB)}^{\text{win}} \tag{4.8}$$

Now, the key thing to note is that all of our adjustments are *multiplications*. Remembering our elementary school rules, remember that multiplication doesn't depend on order. In other words, we can put these *in any order we want* and the result will be the same!

So, let's group these together where the wins are all together, and the losses are all together.

$$m = m_0 \overbrace{(1+rB)(1+rB)(1+rB)}^{\text{wins}} \overbrace{(1-r)(1-r)}^{\text{losses}} \tag{4.9}$$

Now, another thing to notice is that, now that all of our wins and losses are next to each other, we're really performing the *same multiplication* over and over again. Can you think of a way to represent the same multiplication over and over again? That's right—exponents. If we use W to represent all of our wins, and L to represent all of our losses, the equation will look like this:

$$m = m_0 \overbrace{(1+rB)^W}^{\text{all the wins}} \overbrace{(1-r)^L}^{\text{all the losses}} \tag{4.10}$$

4.2.4 How Many Wins and Losses

So now we have a general equation that will work no matter how many wins and losses we have. But how many wins and losses *will* we have, and is there a relationship between them? If you think back to the original thought experiment, we believed that our winner had a 99% certainty of winning. This means that, assuming we are correct, as we play the game over and over and over a lot of times, the percentage of wins will approach 99% (0.99) of the times that we play, and the losses will approach 1% (0.01). So, if we play the game N times, and P is the portion that we will win, then $W = NP$. Likewise, for losses. If we play the game

N times, and P is the portion that we will win, the portion we will lose is $1 - P$. Therefore, $L = N(1 - P)$.

So, we can now rewrite our equation with this in mind:

$$m = m_0(1 + rB)^{NP}(1 - r)^{N(1-P)} \tag{4.11}$$

This is our final equation to model the result of our betting strategies. Our independent variable is r, which, if you recall, tells the portion of our bankroll that we are betting (this is our betting strategy). m is the amount of money we will have at the end, and m_0 is the amount of money we start with (which will be a constant). The rest are constants as well. B is the betting odds, N is the number of times we play the game, and P is the "true" probability of a win.

4.3 Maximizing Results

What we are trying to do is find the value of r in (4.11) which gives us the largest value for m at the end. In calculus, the maximum value is found by taking the derivative and setting it to zero, to see where the slope levels out. The maximum value can also be found at the edges of your interval, but we have already discussed how betting all your money will eventually cause you to lose it all, and betting no money will never gain you anything. So, if there is a maximal strategy, it is somewhere in the middle.

Let's therefore take the derivative of (4.11).

$$m = m_0(1 + rB)^{NP}(1 - r)^{N(1-P)} \tag{4.12}$$

$$\mathrm{d}(m) = \mathrm{d}\left(m_0(1 + rB)^{NP}(1 - r)^{N(1-P)}\right) \tag{4.13}$$

$$\mathrm{d}m = m_0\mathrm{d}\left((1 + rB)^{NP}(1 - r)^{N(1-P)}\right) \tag{4.14}$$

$$\begin{aligned}\mathrm{d}m = m_0 \Big((1 + rB)^{NP}(N(1 - P))(1 - r)^{N(1-P)-1}(-\mathrm{d}r) + \\ (1 - r)^{N(1-P)}(NP)(1 + rB)^{NP-1}(B\,\mathrm{d}r)\Big)\end{aligned} \tag{4.15}$$

$$\begin{aligned}\mathrm{d}m = m_0 \Big(-(1 + rB)^{NP}(N(1 - P))(1 - r)^{N(1-P)-1} + \\ (1 - r)^{N(1-P)}(NP)(1 + rB)^{NP-1}(B)\Big)\ \mathrm{d}r\end{aligned} \tag{4.16}$$

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}r} = m_0 \Big(-(1 + rB)^{NP}(N(1 - P))(1 - r)^{N(1-P)-1} + \\ (1 - r)^{N(1-P)}(NP)(1 + rB)^{NP-1}(B)\Big)\end{aligned} \tag{4.17}$$

Now, that looks ugly, but it will actually simplify quite a bit. Remember, the next step sets the derivative to zero, which we will now do.

$$0 = m_0 \left(-(1+rB)^{NP}(N(1-P))(1-r)^{N(1-P)-1} + (1-r)^{N(1-P)}(NP)(1+rB)^{NP-1}(B) \right) \tag{4.18}$$

The nice thing about being set to zero is that we can now divide by *anything* that's non-zero, and the equation will continue to be true. Also, zero divided by anything is *still zero*, so you can literally get rid of pieces of the equation if you judiciously divide both sides by some component. You have to be a little careful because division can lose solutions, but in this case it isn't a problem.

To start with, let's divide both sides by our starting amount of money, m_0. This yields

$$0 = -(1+rB)^{NP}(N(1-P))(1-r)^{N(1-P)-1} + (1-r)^{N(1-P)}(NP)(1+rB)^{NP-1}(B) \tag{4.19}$$

See? It's just gone. The equation is simpler already. Now, notice that $(1+rB)$ is, in one case, raised to the power of $NP-1$, and, in the other case, raised to the power of NP. If we divide both sides by $(1+rB)^{NP-1}$, then that will remove it entirely from the right-hand side of the addition and remove all the exponents on the left-hand side of the additon, all while keeping the left-hand side still zero.

$$0 = \frac{-(1+rB)^{NP}(N(1-P))(1-r)^{N(1-P)-1} + (1-r)^{N(1-P)}(NP)(1+rB)^{NP-1}(B)}{(1+rB)^{NP-1}} \tag{4.20}$$

$$0 = \frac{-(1+rB)^{NP}(N(1-P))(1-r)^{N(1-P)-1}}{(1+rB)^{NP-1}} + \frac{(1-r)^{N(1-P)}(NP)(1+rB)^{NP-1}(B)}{(1+rB)^{NP-1}} \tag{4.21}$$

$$0 = \frac{-(1+rB)^{\cancel{NP}}(N(1-P))(1-r)^{N(1-P)-1}}{\cancel{(1+rB)^{NP-1}}} + \frac{(1-r)^{N(1-P)}(NP)\cancel{(1+rB)^{NP-1}}(B)}{\cancel{(1+rB)^{NP-1}}} \tag{4.22}$$

$$0 = -(1+rB)(N(1-P))(1-r)^{N(1-P)-1} + (1-r)^{N(1-P)}(NP)(B) \tag{4.23}$$

Now we will do the same disappearing trick with $(1-r)^{N(1-P)-1}$.

$$\frac{0}{(1-r)^{N(1-P)-1}} = \frac{-(1+rB)(N(1-P))(1-r)^{N(1-P)-1}}{(1-r)^{N(1-P)-1}} + \frac{(1-r)^{N(1-P)}(NP)(B)}{(1-r)^{N(1-P)-1}} \tag{4.24}$$

$$0 = -(1+rB)(N(1-P)) + (1-r)(NP)(B) \tag{4.25}$$

Now our equation is *much* simpler! Let's perform a few more simplifications.

$$
\begin{aligned}
0 &= -(1 + rB)(N(1 - P)) + (1 - r)(NP)(B) && (4.26)\\
0 &= -(1 + rB)(N - NP) + (1 - r)(NBP) && (4.27)\\
0 &= -(1 + rB)(N - NP) + NBP - rNBP && (4.28)\\
0 &= -N + NP - rNB + rNPB + NBP - rNBP && (4.29)\\
0 &= -N + NP - rNB + NBP && (4.30)
\end{aligned}
$$

At this point, we can divide everything by N and then solve for r.

$$
\begin{aligned}
0 &= -N + NP - rNB + NBP && (4.31)\\
0 &= -1 + P - rB + BP && (4.32)\\
rB &= BP + P - 1 && (4.33)\\
r &= \frac{BP + P - 1}{B} && (4.34)
\end{aligned}
$$

So, to take our original example, if the betting odds are $\frac{1}{1}$ but the probability is 99% (0.99), then the portion of our bankroll we should bet is

$$
\begin{aligned}
r &= \frac{\frac{1}{1} \cdot 0.99 + 0.99 - 1}{\frac{1}{1}}\\
&= 0.99 + 0.99 - 1\\
&= 0.98
\end{aligned}
$$

So we should bet 98% of our bankroll on this bet to maximize long-term outcomes. So, if we start with \$1,000, our starting bet should be \$980.

(4.34) is known as the Kelly Criterion named after its inventor John Kelly, Jr. This equation, and variations on it, have been foundational for not only managing your bankroll when gambling, but also in other situations where uncertainty plays a large role. For instance, many stock market investors use a Kelly Criterion strategy to manage how much money they invest in any given stock. Since everything has risk, it is unwise to put all your eggs in one basket, and the Kelly Criterion will tell you how big to make your bet in order to maximize your long-term gain.

Variations of the Kelly Criterion are usually done by making alterations to the model equation, (4.11), and then finding the maxima of the new equation. For instance, what if there are more than two outcomes? The equation as we have modeled it only has winning (where you get everything back plus the betting odds) and losing. What if there were other outcomes, as is often the case in investing?

I should point out that while the Kelly Criterion is interesting, and points out some important features of risk-taking, it should not be thought of as the final word in risk allocation. There are a lot of other considerations and constraints that should be considered. For instance, perhaps you are looking for *stability* of return and not the *maximum* return. The Kelly criterion is simply about the long haul, but, if you have short-term constraints, you may want to bet less.

An excellent discussion of the limitations of the Kelly Criterion can be found in Episode 110 of The Bob Murphy Show podcast, available at `https://www.bobmurphyshow.com/110`.

Part II

Biology

Chapter 5

The Logistics Curve

In this chapter we will cover the logistics curve—a growth curve that has limits. While this curve was designed for biology, it actually has lots of uses in many different fields. The growth of a business, the growth of technology—many things follow a logistics curve. Here we will show the thinking behind the curve and how it was constructed.

5.1 Exponential Growth of Biology

In Chapter 3 we talked about the exponential curve. Exponential curves happen when the result of a process leads to more of the process occurring. So, the specific example given was compound interest. The process utilizes money. But the result is *also* money. Therefore, the more the proces is run, the more inputs we have to the process.

The same thing happens with biological reproduction. Reproduction uses organisms, and then produces organisms. Those organisms then become part of the population that produces more organisms.

If I start off with just a pair of organisms—a male and a female—we don't have the kind of clean exponential curve, because the process isn't continuous. If they are humans, then it would take nine months before the first baby came, and more than a decade before that person was capable of participating in human reproduction.

However, as populations grow, they start behaving more and more like continuous curves. While a population of two or ten isn't very continuous, a population of ten thousand operates much more like a continuous curve.

Let us think back to our original model of exponential growth. While previously we were talking about money, now we are talking about population, which we will designate as p. The rate of change of population over time (t) can be understood using the differential $\frac{\mathrm{d}p}{\mathrm{d}t}$. Therefore, if the growth rate is R, then the differential model for this is

$$\frac{\mathrm{d}p}{\mathrm{d}t} = Rp \tag{5.1}$$

In other words, the growth rate *in population* is the growth rate as a percentage of the population multiplied by the current members of the population.

5.2 Limits to Growth

Now, when a population inhabits a new area, they do in fact grow basically in an exponential fashion. However, we know from experience that eventually that exponential growth stops. Otherwise, our homes would be overrun with cute woodland creatures (and the not so cute ones as well).

In the early 1800s, Pierre Verhulst thought about this, and decided that there was a "crowding effect" that was limiting growth. As environments get more crowded, either by lack of food or lack of desire, population growth slows. This idea was further developed Raymond Pearl and Lowell Reed, who, instead of focusing on the crowding effect, said that a given environment had a fixed "carrying capacity" (designated as K) which was the maximum number of individuals that the environment could support. As the environment got more crowded, it decreased the rate at which organisms reproduced. If there by some reason wound up being more organisms than the environment could support, the growth rate would be negative, to bring the population back in line with the carrying capacity.

So let's look again at our rate equation and see if there might be some way to incorporate these ideas into the equation.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = Rp \tag{5.2}$$

What we need is a tuning knob that will adjust this rate based on how close it is to the carrying capacity. For right now, let's call this tuning knob B.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = Rp \cdot B \tag{5.3}$$

So, if B is 1, then we have our normal exponential growth. We want this tuning knob to gradually turn down as we get closer to our carrying capacity. If we are *at* our carrying capacity, we want the tuning knob to be zero (no growth). If we are *beyond* our carrying capacity, we want the tuning knob to be negative.

What we want is to come up with a formula for B that starts near one when the population is small, and goes to zero as the population gets close to the carrying capacity (K).

Let's think about the ratio $\frac{p}{K}$. This is actually the *opposite* of the tuning knob that we want. If p is small, then $\frac{p}{K}$ is zero. If p is close to K, then $\frac{p}{K}$ is close to one. We wanted the opposite, so all we have to do is subtract it from one.

$$B = 1 - \frac{p}{K} \tag{5.4}$$

Now this becomes the tuning knob that we want.

So, we can plug this into (5.3) and get the fundamental differential equation for the logistic curve.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \overbrace{Rp}^{\text{growth rate}} \underbrace{\left(1 - \frac{p}{K}\right)}_{\text{tuning knob}} \tag{5.5}$$

5.3 Integrating the Equation

Integrating the logistic curve takes a little bit of work. Let's start by multiplying through.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = Rp\left(1 - \frac{p}{K}\right) \tag{5.6}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = Rp - \frac{Rp^2}{K} \tag{5.7}$$

This equation makes it pretty easy to separate variables for integration.

$$\frac{\mathrm{d}p}{Rp - \frac{Rp^2}{K}} = \mathrm{d}t \tag{5.8}$$

$$\int \frac{\mathrm{d}p}{Rp - \frac{Rp^2}{K}} = \int \mathrm{d}t \tag{5.9}$$

$$\int \frac{\mathrm{d}p}{Rp - \frac{Rp^2}{K}} = t + C \tag{5.10}$$

The left-hand side can be solved by partial fraction decomposition, which is annoying to perform but straightforward. If you don't remember partial fraction decomposition, you might want to review it before going forward. First we factor the denominator, which factors into p and $R - \frac{R}{K}p$ (remember, R and K are constants for this exercise). Then we presume there is an A and B (not the same B we were using before) which can be used for partial fractions.

$$\frac{1}{(p)\left(R - \frac{R}{K}p\right)} = \frac{A}{p} + \frac{B}{R - \frac{R}{K}p} = \frac{AR - A\frac{R}{K}p + Bp}{(p)\left(R - \frac{R}{K}p\right)} \tag{5.11}$$

Now we can solve for A and B.

$$AR = 1 \tag{5.12}$$

$$A = \frac{1}{R} \tag{5.13}$$

$$A\frac{R}{K}p = Bp \tag{5.14}$$

$$A\frac{R}{K} = B \tag{5.15}$$

$$\frac{1}{R}\frac{R}{K} = B \tag{5.16}$$

$$\frac{1}{K} = B \tag{5.17}$$

So, now we can separate out the factors which will make it much easier to integrate.

$$\int \frac{\mathrm{d}p}{Rp - \frac{Rp^2}{K}} = t + C \tag{5.18}$$

$$\int \frac{\frac{1}{R}\mathrm{d}p}{p} + \int \frac{\frac{1}{K}\mathrm{d}p}{R - \frac{R}{K}p} = t + C \tag{5.19}$$

$$\frac{1}{R}\int \frac{\mathrm{d}p}{p} - \frac{1}{R}\int \frac{-\frac{1}{K}\mathrm{d}p}{1 - \frac{1}{K}p} = t + C \tag{5.20}$$

$$\frac{1}{R}\ln(p) - \frac{1}{R}\ln\left(1 - \frac{1}{K}p\right) = t + C \tag{5.21}$$

The transformation to (5.20) may seem a little strange, but it was done so that the numerator of the second quotient matched the differential of the denominator so that it could easily be integrated using u substitution (which was not shown for brevity's sake).

5.4 Solving for p

At this point we have a solution, but it is a little weirdly arranged. We really want an equation for p in terms of t, so we'll have to do a little work to arrange that. The first thing we can do is to multiply both sides by R to make it a little nicer.

$$\ln(p) - \ln\left(1 - \frac{1}{K}p\right) = Rt + RC \tag{5.22}$$

We can then use logarithm rules to combine our two logarithms.

$$\ln\left(\frac{p}{1 - \frac{1}{K}p}\right) = Rt + RC \tag{5.23}$$

Now, we can get rid of the logarithm by exponentiating.

$$e^{\ln\left(\frac{p}{1-\frac{1}{K}p}\right)} = e^{Rt+RC} \tag{5.24}$$

$$\frac{p}{1-\frac{1}{K}p} = e^{Rt}e^{RC} \tag{5.25}$$

Now it is fairly straightforward to solve for p.

$$\frac{p}{1-\frac{1}{K}p} = e^{Rt}e^{RC} \tag{5.26}$$

$$p = e^{Rt}e^{RC}\left(1-\frac{1}{K}p\right) \tag{5.27}$$

$$p = e^{Rt}e^{RC} - \frac{e^{Rt}e^{RC}}{K}p \tag{5.28}$$

$$p + \frac{e^{Rt}e^{RC}}{K}p = e^{Rt}e^{RC} \tag{5.29}$$

$$p\left(1+\frac{e^{Rt}e^{RC}}{K}\right) = e^{Rt}e^{RC} \tag{5.30}$$

$$p = \frac{e^{Rt}e^{RC}}{1+\frac{e^{Rt}e^{RC}}{K}} \tag{5.31}$$

One simplifying feature that we can do is to replace e^{RC} with just C. We can do this because e is a constant, R is a constant, and C is an unknown constant. Therefore, combining these constants in any way will yield another unknown constant. Therefore, we can simplify the equation.

$$p = \frac{Ce^{Rt}}{1+\frac{Ce^{Rt}}{K}} \tag{5.32}$$

So, we're most of the way there, but now we need to solve for that annoying constant, C. We will do this by setting $t = 0$ and let p_0 represent our population at that time, then solve for C.

$$p_0 = \frac{Ce^{R\cdot 0}}{1 + \frac{Ce^{R\cdot 0}}{K}} \tag{5.33}$$

$$p_0 = \frac{Ce^{0}}{1 + \frac{Ce^{0}}{K}} \tag{5.34}$$

$$p_0 = \frac{C}{1 + \frac{C}{K}} \tag{5.35}$$

$$p_0\left(1 + \frac{C}{K}\right) = C \tag{5.36}$$

$$p_0 + \frac{p_0 C}{K} = C \tag{5.37}$$

$$p_0 = C - \frac{p_0 C}{K} \tag{5.38}$$

$$p_0 = C\left(1 - \frac{p_0}{K}\right) \tag{5.39}$$

$$\frac{p_0}{1 - \frac{p_0}{K}} = C \tag{5.40}$$

Kind of an annoyingly complex value for C, but it works. Now we will plug this back in to (5.32).

$$p = \frac{Ce^{Rt}}{1 + \frac{Ce^{Rt}}{K}} \tag{5.41}$$

$$p = \frac{\frac{p_0}{1-\frac{p_0}{K}}e^{Rt}}{1 + \frac{\frac{p_0}{1-\frac{p_0}{K}}e^{Rt}}{K}} \tag{5.42}$$

5.5 Simplifying the Result

(5.42) is correct, but overly complex. In this section, we will do some transformations to simplify the result.

We will start the simplification process by multiplying by $\frac{K}{K}$.

$$p = \frac{K\frac{p_0}{1-\frac{p_0}{K}}e^{Rt}}{K\left(1 + \frac{\frac{p_0}{1-\frac{p_0}{K}}e^{Rt}}{K}\right)} \tag{5.43}$$

$$p = \frac{K\frac{p_0}{1-\frac{p_0}{K}}e^{Rt}}{K + \frac{p_0}{1-\frac{p_0}{K}}e^{Rt}} \tag{5.44}$$

Now we can simplify by multiplying by $\frac{\frac{1-\frac{p_0}{K}}{p_0}}{\frac{1-\frac{p_0}{K}}{p_0}}$.

$$p = \frac{\frac{1-\frac{p_0}{K}}{p_0} K \frac{p_0}{1-\frac{p_0}{K}} e^{Rt}}{\frac{1-\frac{p_0}{K}}{p_0} \left(K + \frac{p_0}{1-\frac{p_0}{K}} e^{Rt}\right)} \tag{5.45}$$

$$p = \frac{Ke^{Rt}}{\frac{1-\frac{p_0}{K}}{p_0} K + \frac{1-\frac{p_0}{K}}{p_0} \frac{p_0}{1-\frac{p_0}{K}} e^{Rt}} \tag{5.46}$$

$$p = \frac{Ke^{Rt}}{\frac{K-p_0}{p_0} + e^{Rt}} \tag{5.47}$$

$$p = \frac{Ke^{Rt}}{\frac{K}{p_0} - 1 + e^{Rt}} \tag{5.48}$$

Now we will multiply by $\frac{p_0}{p_0}$.

$$p = \frac{p_0}{p_0} \frac{Ke^{Rt}}{\frac{K}{p_0} - 1 + e^{Rt}} \tag{5.49}$$

$$p = \frac{Kp_0e^{Rt}}{K - p_0 + p_0e^{Rt}} \tag{5.50}$$

$$p = \frac{Kp_0e^{Rt}}{K + p_0\left(e^{Rt} - 1\right)} \tag{5.51}$$

(5.51) is the standard form of the logistic equation.

5.6 Reviewing the Work

Figuring out the logistics equation took a bit of work, but now we have an equation which models exponential growth up to a carrying capacity. It should be noted that, as with most equations, this is merely a starting point. Organisms (especially humans) oftentimes modify their environment to improve its carrying capacity. So, in such a situation, K would not be constant, but a variable which increases as a function of time.

Additionally, we modified our rate of growth tuning knob as linear $(1-\frac{p}{K})$. However, the tuning knob could take other forms as well. We tend to opt for linear functions as a first approximation of things, and that makes for a fine starting assumption, but it could be different from that.

In any case, the logistics curve is the standard model for growth up to a carrying capacity. It is used not only for modeling population growth, but also for a number of applications to business. If you are introducing a product and you know the maximum likely size of your market, you can plan for the speed of growth (and reduction of that speed) of your product as a function of time.

If you don't know the size of your market, you can estimate it by finding where you are on the logistic curve. If you are on the bottom of the curve, then getting your growth rate up is the most important thing. If you are nearing the top of the curve, increasing the carrying capacity of the market would be the most important thing.

In all, we know that growth always has limits, and the logistics curve is an excellent model for exponential growth within limited boundaries.

Chapter 6

Predator and Prey (Lotka-Volterra) Equations

In Chapter 5 we discussed biological growth in the face of a limited environment. The environment had a specific carrying capacity which affected the ability of organisms to grow. The logistics curve, which we sketched out in that chapter, was very generic—it did not include any specific reason for why the growth rate was limited, it just assumed that it was.

In this chapter, we will model a more specific limitation to organismal growth—predation. Instead of having an abstract "carrying capacity" in the environment, we will model the effect that predators and prey have on each other, and define a set of equations, known as the Lotka-Volterra equations, which model these relationships. Additionally, we will show that these models apply to many systems beyond even biology.

6.1 Thinking About Predator-Prey Relationships

Before we start modeling predator-prey relationships, we should take a moment to simply think about how they work. Let's say that we have an organism X, which is a predator, and an organism Y, which the predator preys upon. Let us also say, for purposes of simplification, that the growth of organism Y is only limited by predation. So, without introducing the predator, we will use exponential growth (see Chapter 3) as the basic model for how the population of Y will grow.

Now, when we introduce the predator, interactions between predator and prey will cause the population of Y to decline. While Y is declining, as long as X has plenty of food (presumably from feeding off of Y), X will exhibit exponential growth. But, if X is growing exponentially, but the population of Y is *not* (because they are being preyed upon), that means that eventually X is going to run out of food. Not having enough food will inhibit their exponential growth.

Once the population of X starts to decline, then the population of Y can grow again. This allows the growth of the population of X, which starts the cycle over again.

6.2 The Differential Model

The discussion in the previous section gives us an overview of what we should expect, but how do we translate that into a differential model?

We will start by defining variables for the size of each population. We can use x as the population size of organism X (the prey) and y as the population size of organism Y (the predator). We can think of the prey as simply reproducing exponentially. As we saw in (3.2), a differential model for exponential growth can be simply modeled as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax \tag{6.1}$$

for some growth rate A. However, this rate is going to be attenuated by interactions between predator and prey. These interactions are going to be based on the relative population sizes. As there are more predators, there will be more predators searching for prey, and therefore more occasions where they find the prey and eat them. As the number of prey grows, then it will be easier for predators to find prey, and therefore that will also grow the number of occasions where the predator finds and eats prey.

Therefore, we can model these interactions by a term Bxy, where B is some constant. The growth rate will be depressed by these interactions, so we can modify the equation to make the rate

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax - Bxy. \tag{6.2}$$

For the predators, we will assume that their growth rate will be declining in absence of interactions. Therefore, we can begin by modeling their growth as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -Cy. \tag{6.3}$$

However, interactions with prey increase their ability to survive and reproduce. Therefore, we can introduce a term Dxy to represent the positive benefits to reproduction that such interactions bring. The equation can then be modified to take this into account.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Dxy - Cy \tag{6.4}$$

Therefore, the final differential model will be the pair of equations (6.2) and (6.4).

6.3 Solving the Model

Now that we have a model for these equations, we can use this model to create an equation governing the relationship between predator and prey.

The first thing to notice is that the right-hand side of our model equations do not include a term for t. Therefore, if we divide $\frac{\mathrm{d}y}{\mathrm{d}t}$ by $\frac{\mathrm{d}x}{\mathrm{d}t}$ it actually *removes* time as a variable altogether.

$$\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{Dxy - Cy}{Ax - Bxy} \tag{6.5}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}\,\frac{Dx - C}{A - By} \tag{6.6}$$

As you can see, (6.6) removes time as a consideration altogether. Now, to get it ready for integration, we just need to separate out the variables.

$$\frac{A - By}{y}\,\mathrm{d}y = \frac{Dx - C}{x}\,\mathrm{d}x \tag{6.7}$$

$$\frac{A}{y}\,\mathrm{d}y - \frac{By}{y}\,\mathrm{d}y = \frac{Dx}{x}\,\mathrm{d}x - \frac{C}{x}\,\mathrm{d}x \tag{6.8}$$

$$A\frac{\mathrm{d}y}{y} - B\,\mathrm{d}y = D\,\mathrm{d}x - C\,\frac{\mathrm{d}x}{x} \tag{6.9}$$

With the variables separated, we can now integrate both sides of the equation. We will use V as the integration constant since we are already using C for other purposes.

$$\int \left(A\frac{\mathrm{d}y}{y} - B\,\mathrm{d}y\right) = \int \left(D\,\mathrm{d}x - C\,\frac{\mathrm{d}x}{x}\right) \tag{6.10}$$

$$\int A\frac{\mathrm{d}y}{y} - \int B\,\mathrm{d}y = \int D\,\mathrm{d}x - \int C\,\frac{\mathrm{d}x}{x} \tag{6.11}$$

$$A\ln(y) - By = Dx - C\ln(x) + V \tag{6.12}$$

$$V = A\ln(y) - By - Dx + C\ln(x) \tag{6.13}$$

(6.13) is the basic form of the Lotka-Volterra equation, using V as the constant of integration. However, what is V? We can solve for V by replacing x and y with their initial conditions.

Figure 6.1: An Example Predator-Prey Relationship
FIXME: NEED FIGURE HERE (as well as values used)

$$V = A\ln(y_0) - By_0 - Dx_0 + C\ln(x_0) \tag{6.14}$$

We can combine (6.13) and (6.14) to generate a fully-formed equation.

$$A\ln(y_0) - By_0 - Dx_0 + C\ln(x_0) = A\ln(y) - By - Dx + C\ln(x) \tag{6.15}$$

This form shows the full Lotka-Volterra equation in terms of initial conditions x_0 and y_0.

6.4 Interpreting the Equations

Now, remember, we have actually eliminated time from our equation. Therefore, what (6.15) shows is how the populations of each organism relate to one another given the initial conditions. When plotted (depending on the actual constants and initial conditions), this often makes a closed path, which means that the relative populations of predators and prey will fluctuate.

Figure 6.1 shows what this looks like. If you follow the path in a counterclockwise manner, you can see that the growth of the prey (x-axis) eventually allows for the growth of the predators (y-axis), while the growth of the predators eventually causes the population of the prey to decline, which in turn leads to a decline in the predator population.

There is obviously a lot that these equations leave out. What happens if the predator or prey hit other carrying capacity limits in the environment (i.e., Chapter 5)? What about interactions with other species? All of these complicate the model considerably. But, as a basic model of predator/prey interactions, these equations work pretty well.

Also note that there is a real effect not captured by these equations—if the prey actually drops to zero, then so will the predators. Because this is modeled with real numbers, the *graph* can potentially show biological situtations that can't exist, such as having a population size between 0 and 1. According to the graph, the population will recover. According to biology, if the prey population drops too low, then both populations will die off.

6.5 Other Applications of Lotka-Volterra Equations

The Lotka-Volterra equations were actually originally developed in the field of chemistry. In fact, they apply to a great number of systems which combine activators (stimulants for growth) and inhibitors (detractors to growth). They were actually originally developed by Alfred Lotka in chemistry to describe oscillating autocatalytic chemical reactions. Only later were these equations applied to biology.

A more modern application of these equations occurs in economics. Historically, changes in economic output has been considered the result of external shocks/changes to the system. However, by applying Lotka-Volterra dynamics, one can view changes in a system in terms of natural fluctuations within the system itself driven by activators and inhibitors.

For instance, low wages decrease the number of workers willing to work at that wage. However, a decreased number of workers means that the price of workers will increase. Increasing the price of workers will bring more workers into the market, which will again decrease the wages.

As is evident, Lotka-Volterra models have a lot of applications for basic modeling of any system that combines activators and inhibitors.

Chapter 7

The Fundamental Theorem of Natural Selection

7.1 Misc. Notes

FIXME: These are notes that I need to use

Introduction about Darwin, and how people said it wasn't scientific.

I found this passage in Adam Sedgwick's Letter to Professor Owen on March 28, 1860:

"The highest point we can, I think, ever reach is a law of succession of forms, each implying a harmonious reference to an archetype, and each having indications of the action of a final cause—i.e. of intelligent causation, or creation. My belief is: 1st, that Darwin has deserted utterly the inductive track—the narrow but sure track of physical truth,—and taken the broad way of hypothesis, which has led him (spite of his great knowledge) into great delusion; and made him the advocate, instead of the historian—the teacher of error instead of the apostle of truth (361)."

Something from Evolution Wars?

Fundamental Theorem of Natural Selection

Quote from George Price

Quote from Sal Cordova

Look at both Basener/Sanford "The fundamental theorem of natural selection with mutations" and Hoyle's "The mathematics of evolution"

7.2 Fisher's Fundamental Theorem

Fisher's goal was to formulate a mathematical theory of evolution. He wanted to show that natural selection worked essentially as a law, and there were mathematical reasons to think that natural selection was able to evolve populations to greater and greater heights of fitness.

To that end, Fisher developed what he termed the "Fundamental Theory of Natural Selection" (FTNS). With the FTNS, Fisher believed that he had the clenching mathematical argument that had eluded his predecessors.

To understand the FTNS, we have to come up with formal ways of thinking about fitness. There are many ways to understand biological fitness, but one option, which fit the philosophical notions of evolution at the time, was to understand fitness in relation to reproduction. The advantage of relating fitness to reproduction is that one can analyze notions of evolution without "looking under the covers." It gives a measurement without having to argue about whether or not this biological feature is "more fit" than that one, or whether having redundant systems makes you more fit or less fit, etc. The number of offspring you have is an easy measurement to count, and clearly, if you are insufficiently "fit" (by whatever definition), it would be hard to have offspring.

Additionally, Fisher wanted to avoid questions on what the optimum level of reproduction is for a given population. Therefore, he used relative fitness, which is the fitness of an individual *in comparison to* the rest of the population that exists at the same time. So, we aren't comparing a fox to hound dog or a hummingbird to a finch. We are saying, within this particular population, how well is this individual doing *comparatively.*

Also, for a simplifying assumption, the FTNS assumes that there isn't any genetic mixing occurring. For the purpose of the theorem, whatever genetic diversity exists at the start is all you get. There's no sexual reproduction, no horizontal gene transfer, nothing. The FTNS says nothing about how many starting genetic configurations exist, but, as time goes on, they do not change or even cross. Additionally, in the FTNS, the subpopulations don't compete with each other, or even run out of food. There are also no mutations introducing new changes.

To understand the FTNS, imagine that, within your population, there are n subpopulations. Each subpopulation is genetically identical within itself, with no crossing over to other subpopulations. Since there is no competition or environmental factors being considered, the organisms will essentially experience exponential growth (see Chapter 3).

So, if t is time, p_i is the ith subpopulation, and r_i is the growth rate of that subpopulation, then we can model the growth rate of each population using the formula

$$p_i = p_{i_0} e^{r_i t}. \tag{7.1}$$

However, going back to the meaning of this equation, we can also model this as a differential (see (3.2)).

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = r_i\, p_i \tag{7.2}$$

This means that the rate of change of the population (in individual members) is based on the population (p_i) and the rate (r_i). Remember that, since there's no interaction among the populations, and no impact of the environment, the rate will be a constant.

The total population (including all subpopulations) will be

$$p_{\text{total}} = \sum_{i=1}^{n} p_i \tag{7.3}$$

We can then make f_i be the relative frequency of p_i in p_{total}.

$$f_i = \frac{p_i}{p_{\text{total}}} \tag{7.4}$$

Now, since each population i is each growing at a rate of r_i,

Chapter 8

Modeling the Spread of Infectious Diseases

Epidemiology is another area where calculus often comes into play. Modeling the spread of a disease through a population can often be done using differential equations, which can then be solved to yield predictions of what the spread of the disease will look like.

Like any other type of modeling, these models depend on certain assumptions about how the disease spreads, how the population behaves, etc. Additionally, the action of disease in a population is inherently discrete, but we will treat it as continuous in order to simplify the creation and evaluation of the models. As the population under consideration gets bigger, the less distortion the assumption of being continuous gives to the model.

8.1 The SIR Model

One commonly-used epidemiological model is known as the SIR model. This model is known as a "compartmentalized" model because it breaks the population under consideration up into a fixed set of "compartments." In the SIR model, these compartments are S (susceptible individuals—people who are able to get the disease), I (infected individuals—people who currently have the disease), and R (recovered individuals—people who used to have the disease and are now fine).[1] In this basic model, we will assume the disease is non-deadly, and that no one dies for other

[1]Note that while we normally use lowercase letters for variables but here we are using capital letters. The reason is to match with the standards used in epidemiology. Additionally, i can easily be confused with the imaginary unit, so capitalizing it makes it easily distinguishable as a part of the model.

reasons during the period of spread.

So, in this model, at any given time, each person is in only one of the compartments. Therefore, for a population N, we can say that, at all times,

$$N = S + I + R. \tag{8.1}$$

That is, N is all the members of the population, and they must be in exactly one of the compartments.

Now, as people get infected, they will move from S to I, and, once they are no longer spreading the disease, they will be moved to R.[2] The rate at which this move from S to I is happening is known as the "incidence rate," and is equivalent to $-\frac{\mathrm{d}S}{\mathrm{d}t}$. It is negative, because the actual $\mathrm{d}S$ is *negative* because that population is *decreasing* when individuals move from S to I. Time (t) is generally measured in days.

8.2 Modeling Incidence Rates

So what parameters control the incidence rate? If you think about it, what causes a disease to spread is that a member of I comes into close contact with someone from S. If a member of I comes into contact with someone else from I, nothing happens because they already have the disease. If a member I comes into contact with someone from R, they presumably have developed resistance ot the disease, so they are no longer susceptible. Therefore, nothing happens in that case, either. Additionally, nothing happens when someone from S comes into contact with someone from S, or if someone from R comes into contact with someone from S.

Therefore, the incidence rate is going to be largely based on the frequency of contact between members of I and members of S. Because of this, we need to model the contact that I has with S.

Let's say that an average person comes into contact with C people per unit of time (days in our case).[3] The chance that a given person contacted will be in S is $\frac{S}{N}$. Therefore, each unit of time (day) a member of I will be in contact with $C\frac{S}{N}$ people from S.

Additionally, however, not every contact is going to spread the disease. Therefore, we need to have another parameter, which is the probability that a contact will spread the disease. We will call this probability P.

This means that the number of people that a given person in I will spread the disease to during a unit of time (day) can be given by $PC\frac{S}{N}$. This is true for each

[2]We will consider them "recovered" once they are no longer spreading. This way, if we remove them from the population once symptoms are developed, we can have a specific amount of time that they were contagious. This simplifies the model without much loss in expressibility.

[3]In some models, C increases proportionally with the population size, so it is actually CN.

member of I. While it is true that, in real life, there will be some overlap in the people contacted by individual members of I, for purposes of this model, we will ignore this.

Based on these considerations, the incidence rate can be modeled as

$$-\frac{\mathrm{d}S}{\mathrm{d}t} = PC\frac{S}{N}I. \tag{8.2}$$

Now, of these parameters, P, C, and N are all constants. Therefore, they can be combined into a single parameter, which we will call β, so that

$$\beta = \frac{PC}{N}. \tag{8.3}$$

This simplifies (8.2) to be

$$-\frac{\mathrm{d}S}{\mathrm{d}t} = \beta SI. \tag{8.4}$$

8.3 Modeling Recovery Rates

The recovery of individuals is measured by $\frac{\mathrm{d}R}{\mathrm{d}t}$. You might have expected this to be modeled using $\mathrm{d}I$, but, in fact, the change in individuals in I is due to *both* the change in S and the change in R. Therefore, the recovery rate is simply the rate of people which are added to R.

The removal of individuals from the I (infected) into R (recovered or otherwise sufficiently outside of the population to prevent spreading) is often modeled as a simple constant rate, known as α. In other words,

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \alpha I. \tag{8.5}$$

Part III

Geometric Formulas

Chapter 9

The Grazing Goat Problem

Part IV

Physics

Chapter 10

Physics Basics

Chapter 11

The Kinematic Equations

In physics, a set of commonly used equations, known as the "kinematic equations," are used to analyze the movement of objects under common (though idealized) physical conditions such as gravity. The kinematic equations will tell you such things as where a cannonball will land when fired into the air, how to find the height of a building based on how long it takes an object to hit the ground, and how long it will take a coin tossed in the air to hit the ground.

These are known as the "kinematic" equations because they deal with moving bodies, and the Greek word for movement is kinema ($\kappa\iota\nu\eta\mu\alpha$). These equations are usually given in high school physics courses with very little explanation of where they come from. However, as is often the case, they are actually a straightforward application of calculus.

The kinematic equations tell you, presuming you are moving with a *constant* acceleration, a formula to determine what your position will be at any given time. This is especially useful since gravity, the ever-present actor in earth-based physics, shows its effects on objects as a constant acceleration ($-9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$ when near sea level).

The kinematic equations are as follows. Given a constant acceleration A, position p, starting position p_0, and starting velocity v_0, an object's position at any given time (t) is

$$p = \frac{1}{2}At^2 + v_0\,t + p_0. \tag{11.1}$$

The object's velocity at any given time (t) is

$$v = At + v_0. \tag{11.2}$$

While these equations are often given to high-school physics students without much explanation, it turns out that, like many things in math and physics, the formulas actually arise straightforwardly from the *meaning* of position, velocity, and acceleration, with a small application of calculus.

11.1 Position, Velocity, and Acceleration

Oftentimes in mathematics, the most important thing you need to know is the *meanings* of the words being used. As I often say, step one of mathematics is philosophy. In other words, you have to have thought through all of your definitions carefully in order to arrive at a system in which mathematical statements can be applied.

Position is the foundational idea that we need to start with. For the purpose of explanation, we will consider an object's position on a line.[1] That is, the object can only move back and forth on the line, not in any other direction. Oftentimes, we are considering vertical physical positions. In these cases, a positive value will mean "up" and a negative value will mean "down." Because it is the standard unit of measurement, we will mark positions in meters.

Velocity is the measure of how fast an object's position is changing per unit of time. Ideally, we measure the instantaneous velocity, which is the instantaneous rate of change in position with respect to time. In other words, velocity is the derivative of position with respect to time. If v is velocity, p is position, and t is time, then

$$v = \frac{\mathrm{d}p}{\mathrm{d}t}. \tag{11.3}$$

Since position is measured in meters and time is measured in seconds, velocity is measured in meters per second. While velocity is how fast an object's position is changing, an objects acceleration is how fast that object's velocity is changing. In other words, acceleration is the derivative of velocity with respect to time. If a is acceleration, v is velocity, and t is time, then

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}. \tag{11.4}$$

Since velocity is measured in meters per second, acceleration is measured in meters per second per second, which can be written as $\frac{\frac{\mathrm{m}}{\mathrm{s}}}{\mathrm{s}}$ or as $\frac{\mathrm{m}}{\mathrm{s}^2}$. For the kinematic equations, a will actually be a constant value, which we will refer to as A. When we are just considering the acceleration due to gravity, $A = -9.81\frac{\mathrm{m}}{\mathrm{s}^2}$.

11.2 Integrating Acceleration

(11.4) gives us a basic starting point for using calculus. It gives us an equation for the differential of v, but not v itself. Integrating us will allow us to determine v from $\mathrm{d}v$.

[1] The kinematic equations also work with vector-based positions, velocities, and accelerations, but, for simplicity, we will focus on the position on a line so we can just use scalar (single-value) quantities.

Remember, these equations all assume a *constant* acceleration. Therefore, let's integrate (11.4), setting the acceleration to the constant A.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = A \tag{11.5}$$

$$\mathrm{d}v = A\,\mathrm{d}t \tag{11.6}$$

$$\int \mathrm{d}v = \int A\,\mathrm{d}t \tag{11.7}$$

$$\int \mathrm{d}v = A \int \mathrm{d}t \tag{11.8}$$

$$v = At + C \tag{11.9}$$

As usual, integration leaves us with an arbitrary constant, C. However, we can find C in terms of initial conditions. We will set $t = 0$, and then use v_0 to represent velocity at that time.

$$v_0 = A \cdot 0 + C \tag{11.10}$$

$$v_0 = C \tag{11.11}$$

Therefore, C is the same as our starting velocity. We can then insert that into (11.9) and arrive at (11.2).

$$v = At + v_0$$

11.3 Integrating Velocity

To move forward, we need only to recognize that velocity itself is actually a derivative, as noted in (11.3). Therefore, we can rewrite (11.2) to say

$$\frac{\mathrm{d}p}{\mathrm{d}t} = At + v_0. \tag{11.12}$$

What we want to do is convert this equation that includes the differential of p into an equation for p itself. To do that, we can multiply both sides by $\mathrm{d}t$ and integrate (remember, A and v_0 are constant).

$$\frac{\mathrm{d}p}{\mathrm{d}t} = At + v_0 \tag{11.13}$$

$$\mathrm{d}p = At\,\mathrm{d}t + v_0\,\mathrm{d}t \tag{11.14}$$

$$\int \mathrm{d}p = \int (At\,\mathrm{d}t + v_0\,\mathrm{d}t) \tag{11.15}$$

$$\int \mathrm{d}p = \int At\,\mathrm{d}t + \int v_0\,\mathrm{d}t \tag{11.16}$$

$$\int \mathrm{d}p = A \int t\,\mathrm{d}t + v_0 \int \mathrm{d}t \tag{11.17}$$

$$p = \frac{1}{2}At^2 + v_0\,t + C \tag{11.18}$$

Again, we are left with a trailing constant. And, again, we will determine what this constant is in terms of initial conditions. We will use p_0 and v_0 to stand in for our position and velocity at time $t = 0$.

$$p_0 = \frac{1}{2}A(0)^2 + v_0(0) + C \tag{11.19}$$

$$p_0 = 0 + 0 + C \tag{11.20}$$

$$p_0 = C \tag{11.21}$$

Substituting in p_0 for C in (11.18) will then yield (11.1).

$$p = \frac{1}{2}At^2 + v_0\,t + p_0$$

11.4 Using the Equation

An example of how to use the equation is to imagine this scenario. You throw a ball directly up into the air. When it leaves your hand, it is 2 meters from the ground, and the ball is traveling at 10 meters per second upward. Assuming no friction and wind resistance, where will the ball be after 1.5 seconds ($t = 1.5\text{s}$)?

To find the answer, we simply use (11.1). Here, the starting position (p_0) is 2m, the starting velocity (v_0) is $10\frac{\text{m}}{\text{s}}$, and the acceleration ($A$) is gravity, which is $-9.81\frac{\text{m}}{\text{s}^2}$.

$$p = \frac{1}{2}At^2 + v_0\,t + p_0 \tag{11.22}$$

$$p = \frac{1}{2}\left(-9.81\frac{\text{m}}{\text{s}^2}\right)(1.5\text{s})^2 + \left(10\frac{\text{m}}{\text{s}}\right)(1.5\text{s}) + 2\text{m} \tag{11.23}$$

$$p = -11.03625\text{m} + 15\text{m} + 2\text{m} \tag{11.24}$$

$$p = 5.96375 \tag{11.25}$$

You can also solve for time to find out when the ball will be at a certain position, such as on the ground. So, the ball will be on the ground when the position is zero.

So, setting $p = 0$, we can solve using the quadratic formula.[2]

$$p = \frac{1}{2}At^2 + v_0\,t + p_0 \tag{11.26}$$

$$p = \frac{1}{2}\left(-9.81\frac{\text{m}}{\text{s}^2}\right)t^2 + \left(10\frac{\text{m}}{\text{s}}\right)\,t + 2\text{m} \tag{11.27}$$

$$p = -4.905\frac{\text{m}}{\text{s}^2}\,t^2 + 10\frac{\text{m}}{\text{s}}\,t + 2\text{m} \tag{11.28}$$

$$t = \frac{-(10) \pm \sqrt{(10)^2 - 4 \cdot (-4.905)(2)}}{2(-4.905)} \tag{11.29}$$

$$t = \frac{-10 \pm \sqrt{139.24}}{-9.81} \tag{11.30}$$

$$t = \frac{-10 \pm 11.8}{-9.81} \tag{11.31}$$

$$t = \frac{10 \pm 11.8}{9.81} \tag{11.32}$$

$$t \approx 2.222\,\text{sec} \;\;\text{or}\;\; -0.1835 \tag{11.33}$$

Obviously, since we are starting at time $t = 0$, then we need a positive time, so the ball hits the ground after approximately 2.222s.

Additionally, we can use (11.2) to tell us how fast the ball is going when it hits the ground.

$$v = At + v_0 \tag{11.34}$$

$$v \approx \left(-9.81\frac{\text{m}}{\text{s}^2}\right)(2.222\text{s}) + 10\frac{\text{m}}{\text{s}} \tag{11.35}$$

$$v \approx -11.80\frac{\text{m}}{\text{s}} \tag{11.36}$$

Therefore, at the moment before the ball hits the ground, it will be traveling at approximately $-11.80\frac{\text{m}}{\text{s}}$, which is $11.80\frac{\text{m}}{\text{s}}$ downward.[3]

As you can see, the kinematic equations can be used to answer basic questions about motion in situations where constant acceleration is occurring.

11.5 Non-Constant Acceleration

While the ‘official” kinematic equations, (11.1) and (11.2), are based on constant acceleration, the fact is that you can use the *ideas* behind these equations to generate your own equations for other situations. As noted in Section 11.1, the key insight

[2]Note that halfway through we will drop the units to make it easier to read and follow, but if you follow the units carefully you will notice that they do in fact properly cancel out.

[3]If you knew the ball's mass, you could use the information in Chapter 12 to calculate the object's kinetic energy before impact.

is that velocity is the derivative of position with respect to time, and acceleration is the derivative of velocity with respect to time. This fundamental relationship is still true even if the acceleration is non-constant.

If the acceleration is non-constant, then all that has to be done is to replace the constant A in (11.5) with whatever expression gives you the value for acceleration, and then move on from there. For example, let's say that, for some strange situation, the formula for acceleration is

$$a = 5 - t. \tag{11.37}$$

Then, we would develop the formula for velocity using that formula in the place of A.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 5 - t \tag{11.38}$$

$$\mathrm{d}v = 5\,\mathrm{d}t - t\,\mathrm{d}t \tag{11.39}$$

$$\int \mathrm{d}v = \int (5\,\mathrm{d}t - t\,\mathrm{d}t) \tag{11.40}$$

$$\int \mathrm{d}v = \int 5\,\mathrm{d}t - \int t\,\mathrm{d}t \tag{11.41}$$

$$\int \mathrm{d}v = 5\int \mathrm{d}t - \int t\,\mathrm{d}t \tag{11.42}$$

$$v = 5t - \frac{1}{2}t^2 + C \tag{11.43}$$

$$v = 5t - \frac{1}{2}t^2 + v_0 \tag{11.44}$$

That last step is based on recognizing what would happen at time $t = 0$ (C becomes equivalent to our starting velocity).

Next, we could determine the equation for position by, again, noting that velocity is simply the derivative of position with respect to time.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 5t - \frac{1}{2}t^2 + v_0 \tag{11.45}$$

$$\mathrm{d}p = 5t\,\mathrm{d}t - \frac{1}{2}t^2\,\mathrm{d}t + v_0\,\mathrm{d}t \tag{11.46}$$

$$\int \mathrm{d}p = \int \left(5t\,\mathrm{d}t - \frac{1}{2}t^2\,\mathrm{d}t + v_0\,\mathrm{d}t\right) \tag{11.47}$$

$$\int \mathrm{d}p = \int 5t\,\mathrm{d}t - \int \frac{1}{2}t^2\,\mathrm{d}t + \int v_0\,\mathrm{d}t \tag{11.48}$$

$$\int \mathrm{d}p = 5\int t\,\mathrm{d}t - \frac{1}{2}\int t^2\,\mathrm{d}t + v_0\int \mathrm{d}t \tag{11.49}$$

$$p = \frac{5}{2}t^2 - \frac{1}{6}t^3 + v_0\,t + C \tag{11.50}$$

$$p = \frac{5}{2}t^2 - \frac{1}{6}t^3 + v_0\,t + p_0 \tag{11.51}$$

Again, the p_0 comes from considering what C evaluates to at time $t = 0$.

So, as you can see, you can develop your own kinematic equations for whatever type of accelerating environment you are considering. The standard kinematic equations are just often used because they represent standard idealized near-earth conditions. But, once you know the thinking behind them, you can adjust them for whatever your needs are. Additionally, there is no real need to memorize the equations once you know where they come from. They literally fall out of the very definition of acceleration.

Chapter 12

Kinetic Energy

Work is the application of a force across a distance. If I push on a TV with a 2 newton force for 3 meters, I have performed 6 newton-meters of work. Another name for a newton-meter is a **Joule**, abbreviated J.

Energy is the ability to do work, and is measured in precisely the same units. If I have 20J of energy, that means that, at most, I can do 20J of work. Usually there is some amount of loss when energy is applied towards work, so you can think of energy as a maximum on the ability to do work. If you have 20J of energy, you will likely do less than 20J of work, but you certainly won't do more.

Kinetic energy, then, is the amount of energy that a moving object has available to it because of its speed. You can think of it as energy that is *stored* in an object's speed.

Imagine an object A moving along at velocity v_1. If that object impacts another object, it will likely do work on that object (i.e., apply force across a distance). Assuming no other forces are in play, that work comes from object A's velocity and mass. So how do we calculate how much kinetic energy a moving object has?

12.1 Setting Up the Model

The two components that we will start with are an object of mass m moving at velocity v. Since energy is the ability to do work, what we will do is calculate the maximum amount of work this object could possibly do based on its mass and velocity. Given our definition of energy as the ability to do work, that will be equivalent the amount of kinetic energy that this object contains.

The definition of work is

$$w = f \cdot d \tag{12.1}$$

where w is work, f is force, and d is distance.

If an object is not changing mass, then force is equivalent to mass multiplied by acceleration. Since forces are equal and opposite, the force being received will be the negative of the force given. Therefore, we can substitute into our equation $-m \cdot a$ for f.

$$w = -m \cdot a \cdot d. \tag{12.2}$$

The transfer of energy to the new object won't happen instantaneously. Since it is work, it takes place across some amount of distance. Additionally, as it transfers energy its velocity will change. We don't know if the velocity will change at a constant rate or at a variable rate as it transfers energy.

Because of this, we need to think about it in smaller incremental units. Let's go down to the infinitely small changes. We can think about this in terms of differentials. Each differential amount of work is based on the current force multiplied by the differential amount of distance traveled. That yields the equation

$$\mathrm{d}w = -ma \, \mathrm{d}d. \tag{12.3}$$

12.2 Some Interesting Manipulations

The problem with (12.3) is that, in it, work is dependent on mass, acceleration, and distance. However, the only one of those we will actually have directly is mass, which will be constant. However, a few manipulations will actually allow us to do things in the terms that we like.

To start with, acceleration, as we have previously noted is simply another name for $\frac{\mathrm{d}v}{\mathrm{d}t}$. This means that the equation can read

$$\mathrm{d}w = -m \, \frac{\mathrm{d}v}{\mathrm{d}t} \, \mathrm{d}d. \tag{12.4}$$

This helps, as now we have $\mathrm{d}v$ (which is at least in terms of a known variable) instead of a. But we still have $\mathrm{d}d$ and $\mathrm{d}t$. Because multiplication is commutative, we can exchange the places of $\mathrm{d}v$ and $\mathrm{d}d$.

$$\mathrm{d}w = -m \, \frac{\mathrm{d}d}{\mathrm{d}t} \, \mathrm{d}v. \tag{12.5}$$

If you look carefully at (12.5), you might notice something interesting. The fraction $\frac{\mathrm{d}d}{\mathrm{d}t}$ is the change in distance over time (which is the same as the change in position of time). That is another way of saying—velocity!

So, if we replace $\frac{\mathrm{d}d}{\mathrm{d}t}$ with v we get the equation

$$\mathrm{d}w = -m\,v\,\mathrm{d}v. \tag{12.6}$$

Now the equation is completely in terms of mass and velocity (and its differential).

12.3 Integrating the Equation

(12.6) gives the relation in differential form. However, what we are looking for is the totality of the work being done. We want the *sum* of all of the work that is done from the time the object impacts to the time the object has transferred all of its energy to the other object. Since the impacting object will be altering its velocity continually, we can sum this up with a definite integration. But what would the bounds of integration be?

Well, we want to find everything from where the object was at full speed to where the object is stopped (i.e., all of its kinetic energy is gone). Full speed is just the starting speed, $v = v_1$. Stopped is just $v = 0$. Therefore, those are the bounds.

So, the integration will be:

$$w_{\text{total}} = \int_{v=v_1}^{v=0} \mathrm{d}w = \int_{v=v_1}^{v=0} -mv\,\mathrm{d}v \tag{12.7}$$

Because the mass is unchanging, it is a constant, and we can move the $-m$ to the other side of the integral.

$$w_{\text{total}} = \int_{v=v_1}^{v=0} \mathrm{d}w = -m\int_{v=v_1}^{v=0} v\,\mathrm{d}v \tag{12.8}$$

Now we have the integral (and the bounds of integration) on the right-hand side entirely in terms of the variable v and its differentials. Therefore, we can solve it straightforwardly.

$$\begin{aligned}
w_{\text{total}} = \int_{v=v_1}^{v=0} \mathrm{d}w &= -m\int_{v=v_1}^{v=0} v\,\mathrm{d}v && (12.9)\\
&= -m\int v\,\mathrm{d}v\Bigg\|_{v=v_1}^{v=0} && (12.10)\\
&= -m\frac{v^2}{2}\Bigg\|_{v=v_1}^{v=0} && (12.11)\\
&= \left(-m\frac{(0)^2}{2}\right) - \left(-m\frac{v_1^2}{2}\right) && (12.12)\\
&= 0 + \frac{1}{2}mv_1^2 && (12.13)\\
&= \frac{1}{2}mv_1^2 && (12.14)
\end{aligned}$$

Therefore, (12.14) is the equation for the total work that could possibly be done by the object due to its velocity, which is just another way of saying it is the formula for its kinetic energy. Therefore, we can rewrite this as the formula for kinetic energy.

$$E_K = \frac{1}{2}mv^2 \tag{12.15}$$

Note that, although we used velocity as a scalar quantity, the equation (and its derivation) works with velocity vectors as well (using the dot product), and the result remains a scalar quantity.

Chapter 13

Escape Velocity

In this chapter we will consider escape velocity and related formulas. Escape velocity is the velocity that an unpowered object needs in order to completely escape the gravitational field of a large object. Imagine that I throw a rock into the air—it will eventually come back down to earth. However, if I were able to throw the rock sufficiently hard, it could in fact leave earth's gravitational field and not come back. Escape velocity is the velocity that this rock would have to be when it leaves my hand in order to do this.

13.1 Acceleration Due to Gravity

Before we talk about escape velocity, we need to talk about gravity. Generally, in basic physics, we talk about the acceleration due to gravity as being a constant, $-9.81\frac{\mathrm{m}}{\mathrm{s}^2}$. This was the case when developing the equations in Chapter 11.

While this is true enough for basic physics, the fact is that the acceleration due to gravity actually varies depending on how far we are from the center of earth's mass. The value $-9.81\frac{\mathrm{m}}{\mathrm{s}^2}$ is actually for sea level, which is 6.371×10^6m.

Newton's law of universal gravitation tells us how to derive the actual value for gravitational acceleration. Newton's law states that, if we have two masses, m_1 and m_2, and their centers are at a distance of r from each other, then the force due to gravity (F_g) is

$$F_g = G\frac{m_1 m_2}{r^2}. \tag{13.1}$$

Here, the force is given in Newtons, the masses are in kilograms, and the radius is in meters. G is the gravitational constant, which, for these units is

$$G = 6.674 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}. \tag{13.2}$$

Given that the mass of the earth (which we will use as m_2) is 5.972×10^{24}kg, we can calculate the force due to gravity for the mass m_1 as being

$$F_g = G\frac{m_1 m_2}{r^2} = 6.674 \times 10^{-11} \frac{5.972 \times 10^{24} m_2}{\left(6.371 \times 10^6\right)^2} = 9.81953\, m_2. \tag{13.3}$$

Force is mass times acceleration, so if we divide both sides by the mass of the object (m_1), we will see that the acceleration is

$$\frac{F_g}{m_1} = 9.81953 \frac{\mathrm{m}}{\mathrm{s}^2}. \tag{13.4}$$

This says that no matter what the mass is, the force at this particular distance from the earth is $9.81953 \frac{\mathrm{m}}{\mathrm{s}^2}$. This is usually dropped to two or three significant figures, with the constant for near-earth acceleration being $9.8 \frac{\mathrm{m}}{\mathrm{s}^2}$. Additionally, the force given is for both the earth and the object. That is, they attract each other. Therefore, for the object itself, the force is a downward force, and so the acceleration will be negative. This yields the common expression for gravitational acceleration,

$$A_g = -9.8 \frac{\mathrm{m}}{\mathrm{s}^2}. \tag{13.5}$$

However, as you can see from how this is derived, it actually varies based on how far away from the center of the earth we are. Now, for anything near-earth, it is fine. We are sufficiently far from the center of the earth that a few thousand feet either way isn't going to alter the force significantly. But, as the distances go closer to outer space, the force does in fact change significantly.

13.2 Work While Traveling Through Varying Gravitational Force

When force is applied to an object across a distance, we can calculate the work performed.

$$\text{work} = \text{force} \times \text{distance} \tag{13.6}$$

Let us say that we want to know the work required to lift an object 10 meters, and we want to take into account the variations due to gravity. How might we do this?

Since the force due to gravity is continually changing, we will have to consider an infinitely small lift of the object. We will move it across a distance dr. This will require a tiny amount of work, where we will have to counter the force of gravity to lift it.

$$\mathrm{d}w = F_g \times \mathrm{d}r \tag{13.7}$$

We are using dr for the distance, because what we are changing is the distance between the centers of the masses, which we denoted as r previously.

Using (13.1), this becomes

$$\mathrm{d}w = G\frac{m_1\, m_2}{r^2}\,\mathrm{d}r \tag{13.8}$$

If we want to know the *total* work across a specific distance, this can be done by summing up all of the infinitesimal amounts of work done along the way using an integral.

$$\int_{r=r_{\text{start}}}^{r=r_{\text{end}}} \mathrm{d}w = \int_{r=r_{\text{start}}}^{r=r_{\text{end}}} G\frac{m_1\, m_2}{r^2}\,\mathrm{d}r \tag{13.9}$$

Now, if we really think about this, we note that most of these quantities are constant:

- The mass of the earth isn't changing.
- The mass of the object isn't changing.
- The gravitational constant is, well, a constant.

Therefore, since they are all constants, we can move all of these outside of the integral.

$$\int_{r=r_{\text{start}}}^{r=r_{\text{end}}} \mathrm{d}w = G\, m_1\, m_2 \int_{r=r_{\text{start}}}^{r=r_{\text{end}}} r^{-2}\,\mathrm{d}r \tag{13.10}$$

This can then be integrated to find the total work done.

$$\begin{aligned}
W_{\text{total}} &= \int_{r=r_{\text{start}}}^{r=r_{\text{end}}} \mathrm{d}w = G\, m_1\, m_2 \int_{r=r_{\text{start}}}^{r=r_{\text{end}}} r^{-2}\,\mathrm{d}r && (13.11)\\
&= G\, m_1\, m_2 \left(\int r^{-2}\,\mathrm{d}r\right)\Bigg|_{r_{\text{start}}}^{r_{\text{end}}} && (13.12)\\
&= G\, m_1\, m_2 \frac{r^{-1}}{-1}\Bigg|_{r_{\text{start}}}^{r_{\text{end}}} && (13.13)\\
&= -\frac{G\, m_1\, m_2}{r}\Bigg|_{r_{\text{start}}}^{r_{\text{end}}} && (13.14)\\
&= -\frac{G\, m_1\, m_2}{r_{\text{end}}} - -\frac{G\, m_1\, m_2}{r_{\text{start}}} && (13.15)\\
W_{\text{total}} &= \frac{G\, m_1\, m_2}{r_{\text{start}}} - \frac{G\, m_1\, m_2}{r_{\text{end}}} && (13.16)
\end{aligned}$$

So, if I wanted to know the work done by moving a 5kg object from sea level to 1,000m above sea level, then I could plug in those values to work the problem.

$$
\begin{aligned}
W_{\text{total}} &= \frac{G\, m_1\, m_2}{r_{\text{start}}} - \frac{G\, m_1\, m_2}{r_{\text{end}}} && (13.17)\\
&= \frac{6.674 \times 10^{-11} \cdot 5 \cdot 5.972 \times 10^{24}}{6.371 \times 10^{6}} - \frac{6.674 \times 10^{-11} \cdot 5 \cdot 5.972 \times 10^{24}}{6.371 \times 10^{6} + 1,000} && (13.18)\\
&= 49089.9549 \ldots \mathrm{N} \cdot \mathrm{m} && (13.19)\\
&\approx 4.909 \times 10^{4} \mathrm{N} \cdot \mathrm{m} && (13.20)
\end{aligned}
$$

13.3 Going Past the Gravitational Field

Therefore, how much work is required to go beyond earth's gravitational field? Technically, the gravitational field extends forever. Nonetheless, it gets significantly smaller as you go further out. As you get closer to infinitely far away from the earth, the amount of force required drops to infinitesimal amounts. This allows us to calculate how much work *would* be required if we were to move the object the infinite amount required to go forever until we were outside of the earth's gravitational field.

Now, this might sound like it would be difficult to calculate how much work would be required to move an object an infinite distance. But, in fact, if we look at (13.16), we can see that something interesting happens when $r_{\text{end}} = \infty$.

$$
\begin{aligned}
W_{\text{total}} &= \frac{G\, m_1\, m_2}{r_{\text{start}}} - \frac{G\, m_1\, m_2}{r_{\text{end}}} && (13.21)\\
\lim_{r_{\text{end}} \to \infty} W_{\text{total}} &= \frac{G\, m_1\, m_2}{r_{\text{start}}} - \frac{G\, m_1\, m_2}{\infty} && (13.22)\\
&\approx \frac{G\, m_1\, m_2}{r_{\text{start}}} - 0 && (13.23)\\
W_{\text{total to infinity}} &= \frac{G\, m_1\, m_2}{r_{\text{start}}} && (13.24)
\end{aligned}
$$

Now, let us take our 5kg object and figure out how much work is needed to lift it out of the earth's gravitational field.

$$
\begin{aligned}
W_{\text{total to infinity}} &= \frac{G\, m_1\, m_2}{r_{\text{start}}} && (13.25)\\
&= \frac{6.674 \times 10^{-11} \cdot 5 \cdot 5.972 \times 10^{24}}{6.371 \times 10^{6}} && (13.26)\\
&= 3.12801 \ldots \times 10^{8} \mathrm{N} \cdot \mathrm{m} \qquad \approx 3.128 \times 10^{8} \mathrm{N} \cdot \mathrm{m} && (13.27)
\end{aligned}
$$

13.4 Finding Escape Velocity

Now we have the information we need to find the escape velocity. (13.24) tells us the amount of work that needs to be done to move an object past a gravitational field. Kinetic energy is the energy that an object has due to its velocity. From Chapter 12, we found that (12.15) shows that the amount of kinetic energy that an object has is $\frac{1}{2}mv^2$.

Remember that energy is simply the ability to do work. Therefore, if (13.24) tells how much work has to be done to lift an object out of its gravitational pull, and (12.15) tells us how much energy is available for work in an object's velocity, then we can set them equal to each other and solve for v to tell us at what velocity an object has to be traveling to have that much energy. Remember, m_1 is the mass of our object, and m_2 is the mass of the earth (or whatever object's gravitation pull we are fleeing).

$$\frac{1}{2}m_1\,v^2 = \frac{G\,m_1\,m_2}{r_{\text{start}}} \tag{13.28}$$

$$v^2 = \frac{2}{m_1}\,\frac{G\,m_1\,m_2}{r_{\text{start}}} \tag{13.29}$$

$$v^2 = \frac{2\,G\,m_2}{r_{\text{start}}} \tag{13.30}$$

$$v = \sqrt{\frac{2\,G\,m_2}{r_{\text{start}}}} \tag{13.31}$$

Here, (13.31) is the equation for escape velocity.

A few notes on this equation:

- This equation no longer depends on the mass of the escaping object, only the mass of the earth (or other large body).

- This equation does not take into account any other forces (such as friction), just the force due to gravity.

- Also remember that energy is rarely converted perfectly into work. Therefore, it is likely that the escape velocity will need to be faster than this.

So, in the case of our 5kg object, how fast does it need to be moving when it leaves the surface of the earth in order to get out of earth's gravitational pull?

$$v = \sqrt{\frac{2\,G\,m_2}{r_{\text{start}}}} \tag{13.32}$$

$$= \sqrt{\frac{2 \cdot 6.674 \times 10^{-11} \cdot 5.972 \times 10^{24}}{6.371 \times 10^6}} \tag{13.33}$$

$$= 11185.7\ldots \frac{\text{m}}{\text{s}} \tag{13.34}$$

$$\approx 1.119 \times 10^4 \frac{\text{m}}{\text{s}} \tag{13.35}$$

Again, note that the mass of the object was not required for this calculation. So, the escape velocity for *any* object from the surface of the earth is $1.119 \times 10^4 \frac{\text{m}}{\text{s}}$.

Chapter 14

The Rocket Equation

The term "rocket" generally refers to a long, thin vehicle that goes (at least initially) directly skyward. Rockets are propelled by ejecting their part of their mass (i.e., the fuel) through its engines. Thus, as the thrust of the rockets go out of the engines, not only is it moving the rocket, it is making the rocket lighter. This means that the thrust of the rocket will actually cause the rocket itself to accelerate faster as it burns its fuel.

Figure 14.1 shows a basic diagram of the rocket. The rocket consists of the rocket body, the fuel, and an engine. The engine takes the fuel and burns it, creating exhaust. The exhaust comes out of the rocket engine and propels the rocket forward.

The equation that governs this process at the basic level was developed by Konstantin Tsiolkovsky, a Soviet rocket scientist in the late 1800s and early 1900s.

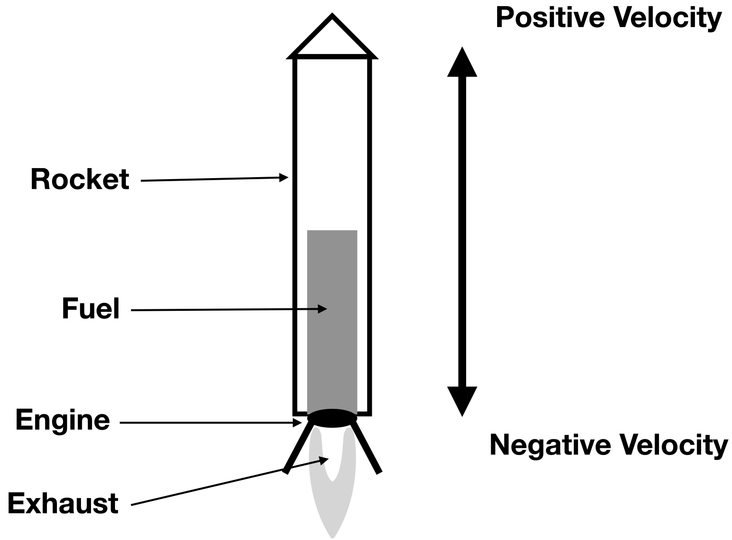
We will be building our rocket equations according to Newtonian physics (i.e., ignoring relativity). We will also assume a stationary frame of reference from which we are measuring velocity.

14.1 Newton's Second Law

The key to understanding Newtonian physics is to recognize the fundamental importance that momentum plays. Newton showed that it was *momentum* that was conserved. This means that if there are no other forces in play, the total of all momentums in a closed system at time $t = t_0$ will be exactly the same as the total of all momentums in the same closed system at time $t = t_1$.

Momentum is calculated as simply being the product of the mass (m) and velocity

Figure 14.1: Parts of a Rocket



(v). Momentums can be added together. So, for any object, you can consider all of the pieces of the object separately, and add their momentums together to get a total momentum. The momentums of each piece may change, but the momentum of the total system *will not*.

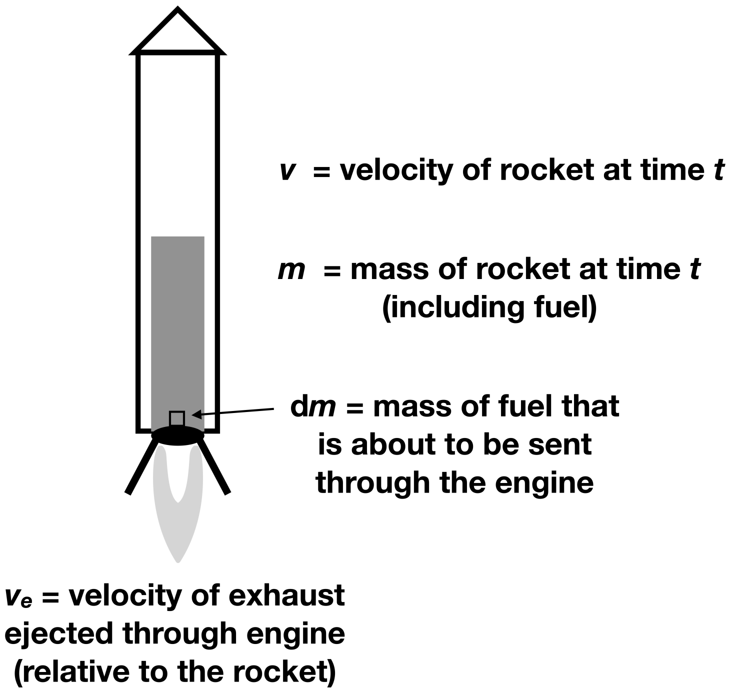
14.2 Building the Rocket Equation Model

The basic model equation that we will use for our rocket will ignore all external forces. That is, we will not be considering gravity or friction. Those can be added in later, but the basic equation ignores these.

The rocket engine converts fuel into exhaust shooting out the back of the rocket. This is what allows a rocket to accelerate through outer space—the rocket is not moved by pushing against anythign, but by exhaust shooting out, and reliance on the conservation of momentum to propel the rocket in the opposite direction.

Figure 14.2 shows the rocket's variables. m will refer to the mass of the rocket and fuel combined at a given point in time. v will refer to the velocity of the rocket at the same time. Note that the mass of the rocket is varying because it is expending fuel, and we hope that the velocity is varying because that is the whole point of using the fuel!

Figure 14.2: Rocket Variables



What the engine does is spew exhaust backwards out of the rocket. The velocity of the exhaust *relative to the rocket* is v_e. This will be a negative value, because it is going in the negative direction. The absolute velocity of the exhaust, however, depends on the velocity of the rocket. Therefore, the absolute velocity of the exhaust is $v + v_e$.

Between two very close moments in time, a bit of fuel mass, which we will designate as $\mathrm{d}m$ will be taken from the rocket and spewed out of the engine. However, because of the law of conservation of momentum, the *total momentum* of the system will not change.

Let's start by looking at the momentum of the system before the fuel is spent. Momemtum is the product of mass and velocity, and the mass of the rocket is m and its velocity is v. We don't care about the previous exhaust, because it is not interacting with the rocket at the present moment. Therefore, the momentum at the first point in time is simply

$$\text{starting momentum} = m\,v. \tag{14.1}$$

Then, a bit of fuel mass, $\mathrm{d}m$, is burned through the engine. This means that part of the mass, $\mathrm{d}m$, is now traveling at a *different* speed: $v + v_e$. In order to preserve the momentum, this will alter the speed of the rocket. The rocket, however, has now changed mass. Its new mass is $m + \mathrm{d}m$. We are using addition because we are moving forward. The *change* $\mathrm{d}m$ will be added (since this is about the change in the variable m), but $\mathrm{d}m$ itself will be negative. We will be adding a negative sign (making it positive) when considering the momentum of $\mathrm{d}m$ in the exhaust.

The new velocity of the rocket is $v + \mathrm{d}v$, where $\mathrm{d}v$ is simply the change in the rocket speed. $\mathrm{d}v$ will wind up being positive, because the expulsion of fuel is driving the rocket's velocity forward.

The total momentum of the system at this point in time can be calculated.

$$\text{momentum at next point in time} = \overbrace{(m + \mathrm{d}m)(v + \mathrm{d}v)}^{\text{momentum of rocket}} + \underbrace{(-\mathrm{d}m)(v + v_e)}_{\text{momentum of exhaust}} \tag{14.2}$$

This can be further expanded and simplified.

$$\begin{aligned}
\text{momentum at next point in time} &= (m + \mathrm{d}m)(v + \mathrm{d}v) + (-\mathrm{d}m)(v + v_e) &(14.3)\\
&= v\,m + v\,\mathrm{d}m + m\,\mathrm{d}v + \mathrm{d}m\,\mathrm{d}v - v\,\mathrm{d}m - v_e\,\mathrm{d}m &(14.4)\\
&= v\,m + m\,\mathrm{d}v + \mathrm{d}m\,\mathrm{d}v - v_e\,\mathrm{d}m &(14.5)
\end{aligned}$$

Now, if a differential is an infinitely small amount, the product of two differentials is infinitely smaller. Therefore, the term $\mathrm{d}m\,\mathrm{d}v$ can be taken as inconsequential and removed.

$$\text{momentum at next point in time} = v\,m + m\,\mathrm{d}v - v_e\,\mathrm{d}m \tag{14.6}$$

The law of conservation of momentum says that, since there were no other forces in play, (14.1) is equal to (14.6), so we can set them equal and simplify.

$$m\,v = v\,m + m\,\mathrm{d}v - v_e\,\mathrm{d}m \tag{14.7}$$

$$0 = m\,\mathrm{d}v - v_e\,\mathrm{d}m \tag{14.8}$$

$$m\,\mathrm{d}v = v_e\,\mathrm{d}m \tag{14.9}$$

(14.9) can have both sides divided by dt to get what is known as the "ideal rocket equation," which is

$$m\,\frac{\mathrm{d}v}{\mathrm{d}t} = v_e\,\frac{\mathrm{d}m}{\mathrm{d}t}. \tag{14.10}$$

Note that if you encounter this equation elsewhere, there may or may not be a negative sign in front of v_e. That depends on whether you are considering that the value of v_e should be negative (since the exhaust velocity is in the opposite direction of the rocket) as we are, or if you are taking v_e to be the absolute value of the exhaust velocity (in which case the formula for the total velocity of $-\mathrm{d}m$ would be $v - v_e$, not $v + v_e$).

14.3 Finding an Equation for Velocity

To find an equation for velocity, we'll start with (14.9) and integrate it. Remember that v_e is a constant that is based on the rocket engine.

$$m\,\mathrm{d}v = v_e\,\mathrm{d}m \tag{14.11}$$

$$\mathrm{d}v = v_e\frac{\mathrm{d}m}{m} \tag{14.12}$$

$$\int \mathrm{d}v = \int \left(v_e\frac{\mathrm{d}m}{m}\right) \tag{14.13}$$

$$\int \mathrm{d}v = v_e\int \frac{\mathrm{d}m}{m} \tag{14.14}$$

$$v = v_e\,\ln(m) + C \tag{14.15}$$

This is all well and good, but what is C? To find an equation for C, we will look at this equation starting at time zero.

$$v_0 = v_e\,\ln(m_0) + C \tag{14.16}$$

$$C = v_0 - v_e\,ln(m_0) \tag{14.17}$$

That can be plugged back into (14.15) and simplified.

$$v = v_e\,\ln(m) + v_0 - v_e\,ln(m_0) \tag{14.18}$$

$$v = v_0 + -v_e\,\ln(m_0) + v_e\,\ln(m) \tag{14.19}$$

$$v = v_0 + -v_e(\ln(m_0) - v_e\,\ln(m)) \tag{14.20}$$

$$v = v_0 + -v_e\,\ln\left(\frac{m_0}{m}\right) \tag{14.21}$$

Therefore, if you know the initial weight and velocity of the rocket as well as the speed at which fuel is ejected, you can determine the rocket's velocity based on its current mass. If you know how fast the engine is using the fuel, you can use that to determine the velocity at any given point in time based on how long it takes to burn a mass of fuel of $m_0 - m$.

Note that in (14.20) we factored out a $-v_e$ instead of just v_e in order to keep our equation more in line with other sources of the rocket equation which consider v_e to be positive.

14.4 Getting Other Forces Involved

(14.21) only deals works in absence of other forces. Normally, however, we have to contend with forces such as gravity and friction. In those cases, we normally return to (14.9) or (14.10) to give a starting point to represent the forces internal to the rocket system, and then add in the external forces that are occurring.

The two most important forces to consider for a rocket are gravity (which can occur near any sufficiently large object) and drag (which is primarily in atmospheres).

To account for these forces, let us take a look back at (14.10).

$$m \frac{\mathrm{d}v}{\mathrm{d}t} = v_e \frac{\mathrm{d}m}{\mathrm{d}t}$$

If you notice, since $\frac{\mathrm{d}v}{\mathrm{d}t}$ (the derivative of velocity with respect to time) is another way of talking about acceleration, this is just "mass times acceleration," which is force. In other words, this is an equation for telling us the force that is occurring due to the ejection of mass from the rocket.

All we need to do to understand how the rocket will behave is to add together *all* of the forces in play, and then solve the resulting differential equation.

FIXME: Need to finish out this section

14.5 What the Rocket Equation Tells Us

FIXME: Need to finish out this section

Chapter 15

Newton's Law of Cooling

Newton's law of cooling states that the rate of temperature change in an object is proportional to the difference in temperature between the object and the environment (assuming that the object itself is not generating heat). Isaac Newton first published his law of cooling anonymously in 1701.

This law is somewhat dated, as new laws have been established which model heat flow across a wider range of extreme temperatures more accurately. However, for the case of "ordinary" temperatures, the law has held up quite well. In fact, it is still used today as one of the mechanisms to determine the time of death of a person.

However, the law only gives the relationship between the rate of change of temperature and the temperature difference. It does not tell us at what points in time the object is at different temperatures. In order to do that, we will need to use calculus to convert the dynamic model into a static model.

15.1 The Differential Model

Newton's law of cooling lends itself naturally to a differential model. Let's look at the law again. "The rate of temperature change in an object is proportional to the difference in temperature between the object and the environment." Let's look at the mathematical objects we have:

- The temperature of an object. Let's call this q (since t is normally used for time). This will be a variable that will change throughout the time under consideration.

- The temperature of the environment. Let's call this q_{E}. This is a constant, since the "environment" is considered sufficiently large so that the object has no effect on the environment's temperature.
- The difference between the temperature of the object and the environment, then, will be $q_{\mathrm{E}} - q$.
- The rate of temperature change in the object. This is a time derivative. We can call this $\frac{\mathrm{d}q}{\mathrm{d}t}$.
- Since we are doing a time derivative, t represents the time. In this case, we will use units of "hours."

So how do we arrange all of these things?

The law says that the rate of change ($\frac{\mathrm{d}q}{\mathrm{d}t}$) is proportional to the temperature difference ($q_{\mathrm{E}} - q$). What does this mean? If two things are proportional, that means that if we put them in ratio with each other (i.e., divide them), there will be some constant of proportionality that they are equal to. In other words,

$$\frac{\frac{\mathrm{d}q}{\mathrm{d}t}}{q_{\mathrm{E}} - q} = K. \tag{15.1}$$

However, this is a little unweildy, so it is normally multiplied by the denominator to become

$$\frac{\mathrm{d}q}{\mathrm{d}t} = K(q_{\mathrm{E}} - q). \tag{15.2}$$

This is our basic differential model that we will work from.

15.2 Integrating

To move from our differential model into a static model, we will need to integrate to get rid of the differentials. However, first we need to separate the variables. We can do this straightforwardly by multiplying both sides by $\mathrm{d}t$ and dividing both sides by $q_{\mathrm{E}} - q$.

$$\frac{\mathrm{d}q}{\mathrm{d}t} = K(q_{\mathrm{E}} - q) \tag{15.3}$$

$$\frac{\mathrm{d}q}{q_{\mathrm{E}} - q} = K\,\mathrm{d}t \tag{15.4}$$

Now we are in a position to integrate both sides. Remember that both K and q_{E} are constants. The integration of the left-hand side occurs with a u-substitution of $u = q_{\mathrm{E}} - q$ and $\mathrm{d}q = -\mathrm{d}u$.

$$\int \frac{\mathrm{d}q}{q_{\mathrm{E}} - q} = \int K\,\mathrm{d}t \tag{15.5}$$

$$-\ln(q_{\mathrm{E}} - q) = Kt + C \tag{15.6}$$

With the basic integration done, we can now solve for q itself.

$$-\ln(q_{\mathrm{E}} - q) = Kt + C \tag{15.7}$$

$$\ln(q_{\mathrm{E}} - q) = -Kt - C \tag{15.8}$$

$$q_{\mathrm{E}} - q = e^{-C}\, e^{-Kt} \tag{15.9}$$

$$-q = -q_{\mathrm{E}} + e^{-C}\, e^{-Kt} \tag{15.10}$$

$$q = q_{\mathrm{E}} - e^{-C}\, e^{-Kt} \tag{15.11}$$

Note, however, that e^{-C} is just a constant transformation of an unknown constant. That means that the result is itself just an unknown constant. Since we don't use C anywhere else in the equation, this can just be rewritten as

$$q = q_{\mathrm{E}} - C\, e^{-Kt} \tag{15.12}$$

Now the question is, what is this unknown constant? We can solve for this constant in terms of initial conditions by setting $t = 0$, and then using q_0 to represent the starting temperature of the object.

$$q = q_{\mathrm{E}} - C\, e^{-Kt} \tag{15.13}$$

$$q_0 = q_{\mathrm{E}} - C\, e^{-K(0)} \tag{15.14}$$

$$q_0 = q_{\mathrm{E}} - C \cdot 1 \tag{15.15}$$

$$q_0 = q_{\mathrm{E}} - C \tag{15.16}$$

$$C = q_{\mathrm{E}} - q_0 \tag{15.17}$$

Now we can substitute this in for C in (15.12).

$$q = q_{\mathrm{E}} - (q_{\mathrm{E}} - q_0)\, e^{-Kt} \tag{15.18}$$

Now, (15.18) is the final form of Newton's law of cooling solve for a static relationship between the temperature of the object and time. Note that the constant of proportionality, K, is still lurking in the equation. This is because each body will have its own K that represents the specific cooling properties of the object.

15.3 Finding Time of Death

As mentioned earlier, Newton's law of cooling can be used in certain stable-temperature environments to determine the time of death of a person. Essentially, a body generates its own heat up until the time of death, after which it cools to match the environment. As long as the body has not cooled all the way down, Newton's law of cooling can be used to determine the time of death.

We know that human bodies are naturally at 98.6°F. Therefore, if we use the current time of measurement as $t = 0$, we can then sove for when $q = 98.6$ to determine the time of death. However, in order to do that, we also need to know the constant of proportionality, K. This quantity, though, is different for each body. So we have to start by finding out the K for this body.

In order to do that, we will need to wait a period of time to see how much the body cools during that period of time. We can then use that information to solve for K.

Let's start by rearranging (15.18) to solve for K.

$$q = q_{\mathrm{E}} - (q_{\mathrm{E}} - q_0)\, e^{-Kt} \tag{15.19}$$
$$q - q_{\mathrm{E}} = -(q_{\mathrm{E}} - q_0)\, e^{-Kt} \tag{15.20}$$
$$-\frac{q - q_{\mathrm{E}}}{q_{\mathrm{E}} - q_0} = e^{-Kt} \tag{15.21}$$
$$\ln\left(-\frac{q - q_{\mathrm{E}}}{q_{\mathrm{E}} - q_0}\right) = -Kt \tag{15.22}$$
$$-\frac{1}{t}\ln\left(-\frac{q - q_{\mathrm{E}}}{q_{\mathrm{E}} - q_0}\right) = K \tag{15.23}$$

Let's say that the temperature of the environment (q_{E}) is 60°F. When we found the person dead, their body's temperature was 80°F. Let's call the time we found the person $t = 0$. This means that $q_0 = 80$.

Let's wait two hours (until $t = 2$) and check the temperature again. Let's say that at this time the temperature is 60°F. This means that when $t = 2$, $q = 60$. We can then plug all of these in to solve for K.

$$K = -\frac{1}{t}\ln\left(-\frac{q - q_{\mathrm{E}}}{q_{\mathrm{E}} - q_0}\right) \tag{15.24}$$
$$K = -\frac{1}{2}\ln\left(-\frac{75 - 60}{60 - 80}\right) \tag{15.25}$$
$$K = 0.14384\ldots \tag{15.26}$$
$$K \approx 0.144 \tag{15.27}$$

Now, we have a K that can work in our equation that matches the body in question.

What we want to do now is to take one of these equations and solve for t. This will allow us to plugin in K, t_0 (which is 80), and q (which is 98.6, the temperature of the living body), and solve for t, which will be a negative number telling us how many hours ago the person died.

We can start from either (15.18) or (15.23) and solve for t. It's probably most straightforward to start from (15.23).

$$K = -\frac{1}{t} \ln\left(-\frac{q - q_{\mathrm{E}}}{q_{\mathrm{E}} - q_0}\right) \tag{15.28}$$

$$t = -\frac{1}{K} \ln\left(-\frac{q - q_{\mathrm{E}}}{q_{\mathrm{E}} - q_0}\right) \tag{15.29}$$

Plugging in the values above yields

$$t = -\frac{1}{K} \ln\left(-\frac{q - q_{\mathrm{E}}}{q_{\mathrm{E}} - q_0}\right) \tag{15.30}$$

$$t = -\frac{1}{0.144} \ln\left(-\frac{98.6 - 60}{60 - 80}\right) \tag{15.31}$$

$$t = -4.56611\ldots \tag{15.32}$$

$$t \approx -4.57 \tag{15.33}$$

This means that the person died approximately 4.57 hours ago.

Part V

Chemistry

Part VI

Electronics

Chapter 16

Basic Electronics

Chapter 17

Calculating RMS Voltage

How much voltage is supplied by the outlets in your walls? How would we even know?

The voltage coming out of your wall is alternating current (AC). That means that sometimes the voltage is positive and sometimes it is negative. In fact, it oscillates between positive and negative voltage at either 50 or 60 hertz (cycles per second), depending on where you live (it is 60 throughout the United States).

So if the voltage is oscillating back and forth from positive to negative, that means that it is never the same voltage. In fact, since it spends as much time being positive as negative, the *average* voltage coming out of the wall is actually zero!

Now, in the United States, we say that the voltage coming out of the wall is 120 volts (120V). What does that mean, though? Is that the absolute value of the peak voltage? Nope. The voltage peaks are at ±170V. Is that the average absolute value of the voltage? No again. That would be 108V. So where does this 120V answer come from?

17.1 RMS Voltage

The most important thing in electronics is not voltage or current, but *power*. Power can be formulated in several ways. If P is power, I is current, and V is voltage, the standard formula (which yields power in watts) is

$$P = V \cdot I. \tag{17.1}$$

However, Ohm's law tells us that voltage can be thought of as current multiplied by resistance. This means that we can write power as

$$P = (I \cdot R) \cdot I = I^2 R. \tag{17.2}$$

Or, in terms of voltages and resistance loads, it can be written as

$$P = \frac{(I \cdot R)(I \cdot R)}{R} = \frac{V^2}{R}. \tag{17.3}$$

This last equation is related to how we determine AC voltage.

The question is, for a given resistance, the power that AC is delivering is equal to what specific voltage that a DC circuit would be delivering? This value is known as the RMS voltage, and the reason for that name will be apparent in a moment.

Note that, for a given resistance, the power in (17.3) is based not on the voltage, but on the *square* of the voltage. Therefore, what we want to know is what is the average of the *squares* of the voltage. Then, when we know that, we can take the square root of it, and that will tell us what the equivalent DC voltage would have been for the same power.

This is known as the RMS voltage, which stands for "root mean squared." In other words, we take the mean (average) of the squares, and then square root the result. This will give us the equivalent DC voltage for the same amount of power.

17.2 Modeling RMS Voltage

When the voltage oscillates between positive and negative, it does so in roughly a sine wave format. The amplitude A of the sine wave is the peak voltage, which in our case is 170V. This can be represented as

$$v = A \sin(t) \tag{17.4}$$

We can put a multiplier in front of t as well, in order to adjust the period of the cycle. However, it doesn't actually matter. We want to know the RMS value for a full cycle—no matter how long it is. Therefore, we will just use $A \sin(t)$ from $t = 0$ to $t = 2\pi$, or one period.

Now, the goal of RMS voltage is to calculate an *average*. In calculus, there is a formula for calculating an average value for a continuous function.

$$\text{average value of } f(x) \text{ from } a \text{ to } b = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a} \tag{17.5}$$

But what do we want to calculate the average of? Remember, we are looking for the *square* of the voltage, not the voltage itself. Additionally, we are calculating across the range of $t = 0$ to $t = 2\pi$. Therefore, we are going to square (17.4) and insert it as the function to average in (17.5).

$$\text{average square of voltage for one cycle} = \frac{\int_0^{2\pi} (A \sin(t))^2 \, \mathrm{d}t}{2\pi - 0} \tag{17.6}$$

However, there is still one more step to go. (17.6) is the "mean square" voltage, but the RMS voltage is the square root of that. Therefore,

$$v_{\mathrm{RMS}} = \sqrt{\frac{\int_0^{2\pi} (A \sin(t))^2 \, \mathrm{d}t}{2\pi}}. \tag{17.7}$$

17.3 Integrating the Equation

Now that we have the model for RMS voltage, we just need to integrate to find out the voltage. The first step of this is to move A all the way out of the integral.

$$v_{\mathrm{RMS}} = \sqrt{\frac{\int_0^{2\pi} (A \sin(t))^2 \, \mathrm{d}t}{2\pi}} \tag{17.8}$$

$$= \sqrt{\frac{\int_0^{2\pi} A^2 \sin^2(t) \, \mathrm{d}t}{2\pi}} \tag{17.9}$$

$$= \sqrt{\frac{A^2 \int_0^{2\pi} \sin^2(t) \, \mathrm{d}t}{2\pi}} \tag{17.10}$$

We can then clean things up to separate out our constants from the integral itself.

$$v_{\mathrm{RMS}} = \sqrt{\frac{A^2}{2\pi} \int_0^{2\pi} \sin^2(t) \, \mathrm{d}t} \tag{17.11}$$

Before we integrate, it is actually much easier to use the trigonometry power-reducing formula for sine to simplify the integrand. The power-reducing formula is

$$\sin^2(t) = \frac{1 - \cos(2t)}{2} \tag{17.12}$$

We can merge this into the integral and simplify.

$$v_{\text{RMS}} = \sqrt{\frac{A^2}{2\pi} \int_0^{2\pi} \sin^2(t)\,\mathrm{d}t} \tag{17.13}$$

$$= \sqrt{\frac{A^2}{2\pi} \int_0^{2\pi} \frac{1 - \cos(2t)}{2}\,\mathrm{d}t} \tag{17.14}$$

$$= \sqrt{\frac{A^2}{4\pi} \int_0^{2\pi} (1 - \cos(2t))\,\mathrm{d}t} \tag{17.15}$$

$$= \sqrt{\frac{A^2}{4\pi} \int_0^{2\pi} (\mathrm{d}t - \cos(2t)\,\mathrm{d}t)} \tag{17.16}$$

Since this is a definite integral, we should separate out the indefinite integral from its evaluation, and then evaluate the integral.

$$v_{\text{RMS}} = \sqrt{\frac{A^2}{4\pi} \int (\mathrm{d}t - \cos(2t)\,\mathrm{d}t)\Big|_0^{2\pi}} \tag{17.17}$$

$$= \sqrt{\frac{A^2}{4\pi} \left(t - \frac{1}{2}\sin(2t)\right)\Big|_0^{2\pi}} \tag{17.18}$$

This can be evaluated and simplified quite a bit.

$$v_{\text{RMS}} = \sqrt{\frac{A^2}{4\pi} \left((2\pi - 0) - \frac{1}{2}\left(\sin(2(2\pi)) - \sin(2(0))\right)\right)} \tag{17.19}$$

$$= \sqrt{\frac{A^2}{4\pi} \left(2\pi - \frac{1}{2}\left(\sin(4\pi) - \sin(0)\right)\right)} \tag{17.20}$$

$$= \sqrt{\frac{A^2}{4\pi} \left(2\pi - \frac{1}{2}(0 - 0)\right)} \tag{17.21}$$

$$= \sqrt{\frac{A^2}{4\pi} (2\pi - 0)} \tag{17.22}$$

$$= \sqrt{\frac{A^2}{4\pi} (2\pi)} \tag{17.23}$$

$$= \sqrt{\frac{A^2}{2}} \tag{17.24}$$

$$= \sqrt{A^2 \frac{1}{2}} \tag{17.25}$$

$$= A\sqrt{\frac{1}{2}} \tag{17.26}$$

So, after a lot of calculation, the RMS voltage winds up being simply the amplitude multiplied by the square root of $\frac{1}{2}$, which is $\sqrt{\frac{1}{2}} = 0.7071\ldots \approx 0.7071$.

The RMS voltage, then, is just the amplitude multiplied by 0.7071. Since the peak voltage is 170, the RMS voltage is $170 \cdot 0.7071 \approx 120.2 \approx 120$V.

And that is why we say that the voltage coming out of your wall outlets is 120V.

Chapter 18

Capacitor-Based Timers

In electronics, timers are often built by putting a resistor in series with a capacitor and waiting for the voltage in the capacitor to build up to a certain level before triggering an action.

FIXME: Need description and also a circuit diagram

18.1 Basic Equations Governing the Circuit

Below, v will refer to the voltage across the capacitor, which will vary. dv will refer to the specific change in this particular voltage. Other voltages will be appropriately subscripted. v_{total} is the voltage on the supply (i.e., the battery voltage), and is a constant. The beginning value of v (at time $t = 0$) is 0.

The fundamental equation that capacitors obey is

$$q = C \cdot v \tag{18.1}$$

where q is the charge stored on the capacitor, v is the voltage across the plates, and C is the capacitance. Note that, as current flows into the capacitor (thus increasing charge), this will cause the voltage across the plates to increase. However, when the voltage across the plates is equal to the supply voltage, no more current can move.

We can also take the time derivative of both sides. This yields the equation

$$\frac{dq}{dt} = C\frac{dv}{dt}. \tag{18.2}$$

Interestingly, the change of charge over time is just the current. The current is represented by I. I is capitalized not because it is constant, but just to distinguish it from the imaginary unit. So we can rewrite (18.2) as

$$I = C\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{18.3}$$

Now let's look at the equation for the resistor. Ohm's law gives us the equation as

$$v_{\text{resistor}} = IR \tag{18.4}$$

where I is the current through the resistor and R is the resistance. Note that since the resistor and capacitor are in series, the *same* amount of current will be flowing through both of them. That means that I, though it will vary, at any given time will be the same for both the resistor and the capacitor.

Also note that the total voltage available is fixed by the battery (or other power source). Since the resistor and capacitor are in series, the voltage across both of them is just the voltage across each one added up, and together they add up to the total voltage.

$$v_{\text{total}} = v_{\text{resistor}} + v \tag{18.5}$$

18.2 Building the Differential Model

We can then use (18.4) to substitute in for v_{resistor}.

$$v_{\text{total}} = IR + v \tag{18.6}$$

We can then solve for I.

$$v_{\text{total}} - v = IR \tag{18.7}$$

$$\frac{v_{\text{total}} - v}{R} = I \tag{18.8}$$

As mentioned, I is the same for both the resistor and the capacitor. Therefore, we can use (18.3) to find its value in terms of the capacitor's voltage and capacitance.

$$\frac{v_{\text{total}} - v}{R} = C\frac{\mathrm{d}v}{\mathrm{d}t} \tag{18.9}$$

Now we have two variables, v and t (though t is only implied), and two differentials, $\mathrm{d}v$ and $\mathrm{d}t$. The rest are constants. Therefore, if we can separate the variables, we can integrate the equation.

18.3 Integrating the Equation

The first step to integrating, then, is to separate the variables.

$$\frac{v_{\text{total}} - v}{R} = C\frac{\mathrm{d}v}{\mathrm{d}t} \tag{18.10}$$

$$\frac{v_{\text{total}} - v}{RC} = \frac{\mathrm{d}v}{\mathrm{d}t} \tag{18.11}$$

$$\frac{\mathrm{d}t}{RC} = \frac{\mathrm{d}v}{v_{\text{total}} - v} \tag{18.12}$$

Now that the variables are separated, we can integrate both sides. We will use K as the constant of integration since C is already in use.

$$\int \frac{\mathrm{d}t}{RC} = \int \frac{\mathrm{d}v}{v_{\text{total}} - v} \tag{18.13}$$

$$\frac{1}{RC}\int \mathrm{d}t = \int \frac{\mathrm{d}v}{v_{\text{total}} - v} \tag{18.14}$$

$$\frac{t}{RC} + K = -\ln(v_{\text{total}} - v) \tag{18.15}$$

$$-\frac{t}{RC} + K = \ln(v_{\text{total}} - v) \tag{18.16}$$

Note that, in the last step, since K is an arbitrary constant, making it negative is irrelevant, so we left it positive for simplicity.

This is a true equation, but we would like to solve it for v. To do that, we will start by exponentiating both sides.

$$e^{-\frac{t}{RC}+K} = v_{\text{total}} - v \tag{18.17}$$

$$e^{-\frac{t}{RC}}e^{K} = v_{\text{total}} - v \tag{18.18}$$

$$Ke^{-\frac{t}{RC}} = v_{\text{total}} - v \tag{18.19}$$

On the last step, since e^K is e raised to an unknown constant, e^K is still an unknown constant. Therefore, since we aren't using K anywhere else, we just used it as the value of e^K. Now we can just move v by itself.

$$v = v_{\text{total}} - K\,e^{-\frac{t}{RC}} \tag{18.20}$$

So what is K? To find out, we will set $t = 0$, and set v to the initial capacitor voltage, which is zero.

$$0 = v_{\text{total}} - K\,e^{-\frac{0}{RC}} \tag{18.21}$$

$$0 = v_{\text{total}} - K\,e^{0} \tag{18.22}$$

$$0 = v_{\text{total}} - K \tag{18.23}$$

$$K = v_{\text{total}} \tag{18.24}$$

Plugging that into (18.20) yields

$$v = v_{\text{total}} - v_{\text{total}}\, e^{-\frac{t}{RC}}. \tag{18.25}$$

This can be rearranged to its final form.

$$v = v_{\text{total}} \left(1 - e^{-\frac{t}{RC}}\right) \tag{18.26}$$

As a reminder, v is the voltage across the capacitor at any given time. As you can see, as t increases, $e^{-\frac{t}{RC}}$ gets closer and closer to 0, which means that the right hand side gets closer and closer to v_{total}, but never fully reaches it.

18.4 The RC Time Constant

Because (18.26) is so complicated, engineers have come up with a shortcut for using it, known as the RC Time Constant. This constant is what you get when you multiply the resistance by the capacitance.

This constant tells you how many seconds it takes for you to fill up the capacitor 63.2% of the way. Why that particular amount? Let's look at (18.26) again.

$$v = v_{\text{total}} \left(1 - e^{-\frac{t}{RC}}\right) \tag{18.27}$$

The expression $e^{-\frac{t}{RC}}$ tells us how empty v is (empty, because we subtract it from 1). If we set $t = RC$, then this expression becomes $e^{-\frac{RC}{RC}} = e^{-1} = \frac{1}{e} \approx 0.368$. Therefore, $1 - 0.368 = 0.632$, or 63.2%. In other words, RC, the multiplication of the resistance and the capacitance, tells us how many seconds it takes to fill the capacitor to 63.2% of the supply voltage.

Multiples of this time constant are also used to good effect. By knowing some "landmark" uses of time constants, you can build circuits without having to dive into nasty equations. For instance 0.7 time constants is roughly half full, 3 time constants is 95% full, and 5 time constants is 99% full.

Part VII

Other Applications